NONASSOCIATIVE GEOMETRY IN REPRESENTATION CATEGORIES OF QUASI-HOPF ALGEBRAS

by

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Abstract

It has been understood that quantum spacetime may be non-geometric in the sense that its phase space algebra is noncommutative and nonassociative. It has therefore been of interest to develop a formalism to describe differential geometry on non-geometric spaces. Many of these spaces would fit naturally as commutative algebra objects in representation categories of triangular quasi-Hopf algebras because they arise as cochain twist deformations of classical manifolds. In this thesis we develop in a systematic fashion a description of differential geometry on commutative algebra objects internal to the representation category of an arbitrary triangular quasi-Hopf algebra. We show how to express well known geometrical concepts such as tensor fields, differential calculi, connections and curvatures in terms of internal homomorphisms using universal categorical constructions in a closed braided monoidal category to capture algebraic properties such as Leibniz rules. This internal description is an invaluable perspective for physics enabling one to construct geometrical quantities with dynamical content and configuration spaces as large as those in the corresponding classical theories. We also provide morphisms which lift connections to tensor products and tensor fields. Working in the simplest setting of trivial vector bundles we obtain explicit expressions for connections and their curvatures on noncommutative and nonassociative vector bundles. We demonstrate how to apply our formalism to the construction of a physically viable action functional for Einstein-Cartan gravity on noncommutative and nonassociative spaces as a step towards understanding the effect of nonassociative deformations of spacetime geometry on models of quantum gravity.

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Contents

Chapter 1

Introduction

The content of this thesis is based on the work published in three papers [34, 37, 35] which were completed during this PhD.

1.1 From closed strings to category theory

The topic of this thesis arose from explorations of the interaction between string theory and noncommutative geometry. String theory is widely believed to provide a consistent quantization of general relativity but noncommutative geometry, as a target space approach to quantum gravity, is a very compelling mathematical technique which lends itself to rigorous and abstract mathematical manipulation. The part of this interaction in focus for this thesis and wherein lies its main contribution is on the side of noncommutative geometry.

One can reconstruct a compact Hausdorff space from the commutative algebra of functions on it; this is the content of the Gelfand-Naimark theorem [38]. If one quantises the algebra to a noncommutative one, then one imagines that the noncommutative algebra can be used to reconstruct a noncommutative quantum space. This is idea behind noncommutative geometry. When one is now interested to know geometrical data of the compact Hausdorff space, assuming it is a manifold, one considers vector bundles over the manifold such as its tangent bundle and bundle of one-forms for example. The sections of these vector bundles are finitely generated projective modules over the commutative algebra of functions on the manifold and one can reconstruct the vector bundles from this data; this is the content of the Serre-Swan theorem [63], [62]. If now the function algebra has been quantised to a noncommutative algebra one can imagine that the noncommutative finitely generated projective modules over this noncommutative algebra correspond to noncommutative vector bundles over the corresponding noncommutative quantum space. In noncommutative geometry one accepts the noncommutative algebra as corresponding to a valid noncommutative quantum space and develops in an algebraic way a theory of differential geometry on the noncommutative algebra with the finitely generated projective modules over it. There is no unique way to describe a theory of noncommutative geometry. Different possibilities have been explored in [23, 54, 33, 32, 51]. The approach in this thesis builds on the approach taken in [2, 5] (a pedagogical introduction can be found in [4]) which was subsequently taken forward in [6, 61, 60]. However it attempts to fit this approach into a more abstract framework so that formulae can be motivated in a more axiomatic and principled way. The tools of category theory are well-suited to such an axiomatic and principled approach and are the tools used in this thesis.

The specific topic of this thesis arose from investigations of the interaction between string theory and noncommutative geometry in the context of flux compactifications of closed string theory and the discovery that there is a more intricate geometric structure involved in this case: Closed strings propagating and winding in certain toy-model spacetime backgrounds related by T-dualities to (geometric) flux compactifications of string theory (where the term flux refers to the three form $H = \mathrm{d}B$ where B is the two-form B-field of string theory) probe a noncommutative *and* nonassociative deformation of the phase space geometry. The corresponding spacetime backgrounds are called R-flux compactifications and are referred to as non-geometric spaces as there is no ordinary notion of transition function to glue local trivialisations (cf. e.g. [57, 19, 48, 17, 25, 18, 21, 15]).

In order to understand the effect of these noncommutative and nonassociative deformations on models of quantum gravity it is imperative to develop a description of differential geometry on such non-geometric spaces. An understanding of how a formulation of differential geometry on a phase space may descend to a meaningful theory on quantum spacetime itself could ultimately be found in the context of doubled geometry or double field theory (cf. [18, 28, 41]). Other possibilities that have been explored include the use of membrane sigma models [25] and matrix theory compactifications [21], Work (in [27]) leading up to this thesis showed that the noncommutativity and nonassociativity of flux compactifications of closed string theory can be elucidated in the theory of representations of triangular quasi-Hopf algebras which arise as cochain twist deformations of cocommutative Hopf algebras. This theory, also referred to as twist deformation quantisation, was described in [64] and by Drinfel'd in [30, 31]. It was shown in particular that one could realise the relations for the phase space of the R-flux compactification if one used a star product in the bracket and that this star product comes from a cochain twist $F \in H \otimes H$ of a certain cocommutative Hopf algebra H. The phase space of the R-flux compactification was moreover shown to be a commutative algebra object in the representation category of the quasi-Hopf algebra H_F which arises as the cochain twist of H.

The theory of representations of a triangular Hopf-algebra has been employed in e.g. [6, 13] to understand the effect of noncommutative deformations of geometry. In order to formulate a description of the more intricate *nonassociative* geometry of the R-flux compactification, a more general encompassing mathematical framework than that required for noncommutative geometry is needed. The suitable generalisation to this framework is the theory of representations of *quasi-Hopf* algebras.

The framework provided by category theory for the theory of representations of quasi-Hopf algebras developed in for e.g. [52] then becomes very useful: Cochain twisting defines a functor between closed braided monoidal categories of representations of quasi-Hopf algebras related by a cochain twist. This enables one to place not only the noncommutative and nonassociative algebra of closed string flux compactifications but also its bimodules as commutative and associative objects in a certain closed braided monoidal category.

The problem then becomes one of extracting the abstract principles behind notions of classical differential geometry and formulating them in the framework of category theory. Writing out the formulae explicitly on elements of objects in the category then gives the desired noncommutative and nonassociative geometrical tools. The categorical formalism enables one to make structurally correct definitions for these tools.

A crucial insight at this point is that notions of geometry ought also to be representations of triangular quasi-Hopf algebras. This is because the triangular quasi-Hopf algebra H_F corresponds to the symmetries of the noncommutative and nonassociative space which is an algebra object in the representation category of H_F. Notions of geometry built on these algebra objects ought to be able to transform nontrivially under the action of the symmetries H_F. Then, since notions of differential geometry are universal in the sense that we speak of 'the Leibniz rule' for example, we can express geometrical concepts in terms of universal constructions internal to a closed braided monoidal category.

As representations of a quasi-Hopf algebra these notions of geometry are subject to twist deformation quantisation. Since twist deformation quantisation gives an equivalence between the representation categories of cochain twist related quasi-Hopf algebras, this means that the configurations spaces of geometrical quantities in cochain twist related quasi-Hopf algebras are isomorphic. This solves the problem of quantum rigidity, which is the phenomenon that configuration spaces of geometrical quantities in quantum theories are in general much smaller than their classical counterparts. It is critical to observe at this point that although the quantisation functor gives an isomorphism between configuration spaces, the criteria by which one selects the critical points of actions describing models of physics based on the configuration spaces differs in a significant way. In other words this isomorphism does *not* correspond to a symmetry of the physical theory.

Exploring the syntax of category theory leads to several insightful reformulations of notions of differential geometry by capturing Leibniz rules, quotients and fibered products elegantly. In this way we develop in an abstract fashion a theory of noncommutative and nonassociative geometry of the type required for flux compactifications of closed string theory.

Since the framework is completely general, one is able to apply it to any space which arises as a cochain twist deformation of a classical manifold. Spaces which are noncommutative but strictly associative are also accounted for in this framework by restricting to quasi-Hopf algebras with trivial associator. The restricted framework reproduces results developed in previous work on noncommutative spaces (see e.g. [6]). Noteworthy examples to which this framework applies include the Moyal-Weyl plane, the noncommutative torus, the Connes-Landi spheres (see e.g. [24]) and the Q-flux compactification of closed string theory (which is a noncommutative space which arises after two successive T-duality transformations in the chain of T-dualities leading to the R-flux compactification (see e.g. [27])). These are noncommutative but strictly associative spaces. Our motivating example of R-flux compactifications of closed string theory of course also fits into this framework as both a noncommutative and nonassociative space.

There are several positive spinoffs that arise from the use of category theory. One spinoff is that one does not need to check properties of geometric entities; these are incorporated in the definitions. A second spinoff is that the existence of noncommutative and nonassociative geometry of the type we are considering is guaranteed by our constructions: The cochain twisting functor is found to be an equivalence of closed braided monoidal categories which means that geometric entities on non-geometric flux compactifications are built out of those on classical backgrounds in a structure preserving way. That is the constructions are functorial and therefore noncommutative and nonassociative geometry is immediately guaranteed to exist if the corresponding entities exist on the classical manifolds from which they arise as a cochain twist. As noted before, this does *not* correspond to an equivalence of physical models described by entities in these equivalent categories. Another important spinoff is that from this abstract perspective one is able to solve technical issues in noncommutative geometry such as finding the axiomatically correct construction of lifts of bimodule connections to tensor products.

The main aim of this thesis is then to systematically develop a formalism of noncommutative and nonassociative differential geometry within the framework of the theory of representation categories of quasi-Hopf algebras.

The approach we take to developing this toolkit of differential geometry relies on the programme of noncommutative geometry which extends the Gelfand-Naimark duality between compact Hausdorff topological spaces and the commutative C^*-

algebras of functions on them [38] to noncommutative, and in this case also nonassociative, algebras. Furthermore, it relies on a noncommutative and nonassociative extension of the Serre-Swan theorem [63],[62] which asserts a duality between vector bundles over a manifold and the finitely generated projective bimodules of sections of these vector bundles over the algebra of functions on the manifold.

The main examples to which this formalism applies come as cochain twist deformations of classical differential geometry. Therefore the aim of this thesis is also to develop the theory of twist deformation quantization of all structures involved. The role of twist deformation quantization is however simply to confirm existence of geometrical identities and hence it is sufficient to consider the representation category of an arbitrary quasi-Hopf algebra and develop notions of geometry on one algebra object and its bimodule objects internal to such a representation category.

There is a physical motivation and a mathematical motivation behind this work. The physical motivation is to systematically develop recent observations in string theory which suggest that stringy quantum geometry involves more complicated noncommutative structures than those previously encountered. In particular that quantum spacetime is not only noncommutative but also nonassociative. The mathematical motivation is to address some internal technical issues in noncommutative geometry involving constructions of connections and their lifts to tensor products.

Let us begin by briefly summarizing some of the physical and mathematical background behind these problems.

1.1.1 Non-geometric string theory

The main physical inspiration behind this work was sparked by the recent observations from closed string theory that certain non-geometric flux compactifications experience a nonassociative deformation of the spacetime geometry [57, 19, 48, 17, 25, 18, 21, 15]. We refer to [49, 59, 26, 16] for brief reviews and further references of the aspects of non-geometric string theory discussed below.

Switching on a nonzero magnetic flux in the extra dimensions of closed string theory, i.e. in flux compactifications of *closed* string theory, leads to deformations of the geometry of spacetime: Starting from a geometric frame wherein closed strings propagate in an H-flux background, after three successive T-duality transformations one is led into a non-geometric frame in the sense that coordinates and their duals, together with momentum and winding modes, become intertwined and hence do not permit transition functions between local trivialisations.

It was found by [19, 48] through explicit string theory calculations that closed strings which wind and propagate in these non-geometric backgrounds probe a non-commutative and nonassociative deformation of the spacetime geometry, with deformation parameter determined by the non-geometric flux that arises as the third T-duality transform of the geometric 3-form H-flux.

This property of string geometry was subsequently confirmed by conformal field theory calculations [17, 20] where the non-geometry finds a concrete interpretation: In string theory a geometric spacetime emerges from the left-right symmetric conformal field theory on the closed string worldsheet, whereas T-duality is a left-right asymmetric transformation leading to asymmetric conformal field theories which do not correspond to any geometric target space. In this non-geometric regime of closed string theory the low-energy dynamics is then expected to be governed by a noncommutative and nonassociative theory of gravity.

The physical origins underlying this nonassociative deformation have been elucidated in various ways: by regarding closed strings as boundary excitations of more fundamental membrane degrees of freedom in the non-geometric frame [25], in terms of matrix theory compactifications [21], and in double field theory [18]; they may be connected to the Abelian gerbes underlying the generalized manifolds in double geometry [28, 41].

Nonassociativity in this setting may be encoded by certain triproducts of fields on configuration space predicted by off-shell amplitudes in conformal field theory [17] and in double field theory [18], or by nonassociative $\star$-products from deformation quantization of twisted Poisson structures in the phase space formulation of nonassociative R-space [25] and by integrating higher Lie algebra structures [25, 9]; the equivalence between these two approaches was demonstrated and extended in [7]. A general treatment of nonassociative $\star$-products in this context can be found in [46]. Deformation quantisation was developed in [11], [12]; [65] provides a good introduction.

In [27] it was observed that these nonassociative star products can be alternatively obtained via a particular cochain twisting of the universal enveloping algebra of a certain Lie algebra to a quasi-Hopf algebra. In [27, 7] they verified that the corresponding nonassociative algebras and their basic geometric structures can be obtained by cochain twist quantization, and hence are commutative and associative quantities when regarded as objects in a suitable braided monoidal category. This is the starting point of the work of this thesis.

The cochain twist deformation quantization techniques originally developed by [27] were motivated by the search for a systematic way to generalize notions of differential geometry to non-geometric backgrounds, and in particular to construct nonassociative deformations of field theory and ultimately gravity (see also [7]). This approach is different in spirit to the nonassociative twist deformation of the geometric f-flux frame considered in [29], which does not seem to be of relevance for non-geometric string theory. It does not agree with the string theory inspired nonassociative torus bundles of [57, 40] either, which reproduce the classical limit only up to Morita equivalence. Physically consistent models with novel properties in the context of quantum mechanics were constructed in [27] using this formalism, and of Euclidean scalar quantum field theory in [55].

In order to extend these considerations to more complicated field theories, it was desirable to develop a general systematic formalism for differential geometry on noncommutative and nonassociative spaces internal to the representation category of any quasi-Hopf algebra.

This is the contribution of this thesis and generalizes and extends earlier work [6, 13, 58].

1.1.2 Nonassociative geometry

The study of noncommutative and nonassociative geometry has been referred to as nonassociative geometry in the literature. This study lies under the mathematically established field of noncommutative geometry.

Noncommutative geometry extends the usual duality between compact Hausdorff topological spaces and the commutative C^*-algebras of functions on them (Gelfand-Naimark theorem [38]) and between vector bundles over a manifold and the finitely generated projective modules of sections of these vector bundles over the algebra of functions on the manifold (Serre-Swan theorem [63], [62]) to noncommutative algebraic structures. The idea is to encode the geometrical content of a space in the language of algebra and then generalise the algebraic structures to noncommutative ones (cf. [47, 51, 53]).

The usual approach to noncommutative geometry is to replace all products by noncommutative $\star$-products as is done for example in [3]. Although this is often the correct thing to do, it could yield formulae which do not satisfy axiomatically substantiated properties. The formalism developed in this thesis provides a foundation upon which one may build formulae which satisfy properties motivated by abstract principles.

Elements of differential geometry on classical manifolds may be abstracted to fit into the framework of a closed braided monoidal category. The infinitesimal diffeomorphisms on a classical manifold form a Hopf algebra and act on the commutative algebra of functions on the manifold and on sections of vector bundles such as the tangent bundle over the manifold in an equivariant way. Hence the elements of classical differential geometry are representations of a triangular Hopf algebra. A triangular Hopf algebra is a special case of the more general notion of triangular quasi-Hopf algebra, and cochain twists based on quasi-Hopf algebras can be used to transform one quasi-Hopf algebra into another quasi-Hopf algebra. It is a result (first shown by Drinfel'd in [31] and in [39] for a subcategory of left modules over an algebra object) that the representation categories of cochain twist related quasi-

Hopf algebras are equivalent. This is a very convenient mathematical fact which enables one to translate mathematical structures built in one of these representation categories into another one which is related to it by a cochain twist. On the other hand, physical models built out of the tools found in equivalent categories do not describe the same physical system as the rules by which the physical models are constructed from the tools differs according to the category.

In this thesis we shall show how to build noncommutative and nonassociative tools of geometry in the representation category of a quasi-Hopf algebra using only intuition from classical differential geometry and the machinery of twist deformation quantisation.

1.1.3 Noncommutative connections on bimodules

The notion of connection in noncommutative geometry was first introduced by Connes in [22] in the mid 1980s. Since then they have been investigated further by amongst others [54, 33, 32, 51].

Given a differential calculus over a noncommutative algebra A, one can develop in a purely algebraic fashion a theory of connections on left *or* right A-modules, see e.g. [47] for an introduction. Given an A-bimodule V we may "forget" about its left A-module structure and introduce connections on V as if it were just a right A-module. The problem with taking right A-module connections on A-bimodules is that there is in general no procedure to construct from a pair of such connections on A-bimodules V, W a connection on the tensor product A-bimodule $V \otimes_A W$. The possibility to induce connections to tensor products of A-bimodules is an inevitable construction in noncommutative differential geometry, required for the construction of tensor fields in noncommutative gravity for example. To gain insight into how to solve this problem concerning tensor products of right A-module connections on A-bimodules, we note that there is an analogue in the theory of module homomorphisms: Given two A-bimodule morphisms $f : V \to X$ and $g : W \to Y$, one can take their tensor product as linear maps $f \otimes g$ and induce an A-bimodule morphism $f \otimes_A g : V \otimes_A W \to X \otimes_A Y$ which descends to the quotient of equivalence classes of the tensor product $\otimes_A$. However if f and g are only right A-module morphisms, then there is in general no procedure to construct from this data a right A-module morphism $V \otimes_A W \to X \otimes_A Y$. The problem lies in showing the equivalence of elements of the form $f(v\,a) \otimes_A g(w)$ and $f(v) \otimes_A g(a\,w)$ for $v \in V, a \in A$ and $w \in W$ since g is not left A-linear.

To overcome this problem, the notion of *bimodule connections* was developed in [54, 33, 32]. To define bimodule connections on an A-bimodule V one needs the additional datum of an A-bimodule morphism $\Omega^1 \otimes_A V \to V \otimes_A \Omega^1$, where Ω^1 is the A-bimodule of 1-forms. Given two A-bimodules V, W together with bimodule connections one can construct a bimodule connection on $V \otimes_A W$. From this construction one obtains a bimodule connection on arbitrary tensor products $V_1 \otimes_A \cdots \otimes_A V_n$ of A-bimodules from the choice of a bimodule connection on each component V_i. Although bimodule connections are by now regarded as the standard choice in most treatments of noncommutative differential geometry (see e.g. [13, 14]), there is a drawback with this concept: We notice that the set of all bimodule connections on an A-bimodule V forms an affine space over the linear space of A-bimodule morphisms $V \to V \otimes_A \Omega^1$; this linear space is very small for many standard examples of noncommutative spaces A, so that generally there are not many bimodule connections. For example, if $V = A^n$ and $\Omega^1 = A^m$ are free A-bimodules, then $V \otimes_A \Omega^1 \simeq A^{n\,m}$ and the A-bimodule morphisms $V \to V \otimes_A \Omega^1$ are in one-to-one correspondence with $n \times (n\,m)$-matrices with entries valued in the center of A. Taking the specific example where A is the polynomial algebra of the Moyal-Weyl space $\mathbb{R}^{2k}_\Theta$, then the bimodule connections on $V = A^n$ are parametrized by the *finite-dimensional* linear space $\mathbb{C}^{2n^2\,k}$ because the center of A is isomorphic to $\mathbb{C}$. As a consequence, noncommutative gauge and gravity theories based on the concept of bimodule connections would in general not provide an adequate description of physics as the space of field configurations in this case is too small.

The question now arises whether the conditions on bimodule connections can be weakened in such a way that one can still induce connections to tensor products. A negative answer to this question was given in [42, Appendix A], where it was shown that in a generic situation the existence of the tensor product connection is equivalent to requiring that the individual connections are bimodule connections. It would then appear that there is no way around the concept of bimodule connections in the case where the algebra A and the bimodules V are generic. However, if one restricts to certain classes of algebras and bimodules, namely those which are commutative up to a braiding, then a weaker notion of bimodule connection exists; this weaker notion of bimodule connection was developed in [6] (see also [61, 1] for brief summaries): Given any quasitriangular Hopf algebra H, one considers algebras and bimodules on which there is an action of the Hopf algebra. As H is quasitriangular, i.e. it has an R-matrix, we can restrict to those algebras A for which the product is compatible with the braiding determined by the R-matrix; we call these algebras *braided commutative*. Similarly, we can restrict to those A-bimodules for which the left and right A-actions are identified via the braiding; we call such bimodules *symmetric*. In this setting one can prove that *any* pair of right module connections on V, W induces a right module connection on $V \otimes_A W$. Many examples fit into the formalism developed in [6]: First of all any ordinary manifold M and natural vector bundle $E \to M$ give rise to the algebra $A = C^\infty(M)$ and the A-bimodule $V = \Gamma^\infty(E \to M)$, which satisfy the requirements of braided commutativity and symmetry with trivial R-matrix. (The Hopf algebra H here can be taken to be the universal enveloping algebra of the Lie algebra of vector fields on M.) Furthermore, deformations by *Drinfel'd twists* based on H preserve the braided commutativity and symmetry properties, and hence give rise to noncommutative algebras and bimodules which fit into this framework; the standard noncommutative tori, and more generally the toric noncommutative manifolds (or isospectral deformations) in the sense of [24] of which the Moyal-Weyl space together with its bimodules of vector fields and one-forms are explicit examples of this. Also, the phase space formulation for the nonassociative deformations of geometry that arise in non-geometric R-flux backgrounds of string theory [27] fits into this framework.

The first aim of this thesis is to place the formalism developed in [6] on a rigorous abstract foundation in order to be able to be able to generalise the formalism to nonassociative structures. To this end, the main starting input for this thesis is the observation that the weaker notion of bimodule connection developed in [6] would be described as an internal homomorphism in the representation category $[H, \mathscr{M}]$ of a triangular Hopf algebra H if the monoidal structure in this representation category admits an internal hom-functor, i.e. if $[H, \mathscr{M}]$ is a closed monoidal category. The proof of the latter and the fleshing out of the details of the theory of the closed monoidal category $[H, \mathscr{M}]$, developed in [52], is the content of Chapter 2.

1.1.4 Internal homomorphisms

Internal homomorphisms play a central role in this thesis. They are the maps by which all considered notions of geometry are modelled. To understand their significance we need to consider how the properties of these maps compare to the other maps present in the category, the morphisms. In the context of differential geometry on a manifold and twist quantisations thereof, the quasi-Hopf algebra present is the Lie algebra of infinitesimal diffeomorphisms which act via Lie derivatives on the function algebras and sections of vector bundles of the manifold. The representation category of this quasi-Hopf algebra of infinitesimal diffeomorphisms has as morphisms maps which are equivariant with respect to the action of the infinitesimal diffeomorphisms. Internal homomorphisms on the other hand are not required to preserve the action of infinitesimal diffeomorphisms but instead can be acted upon by them in the adjoint representation. For dynamical fields, notions of geometry ought to be able to transform nontrivially under the action of the infinitesimal diffeomorphisms and therefore be modelled as internal homomorphisms in the category.

Furthermore, when the objects in the representation category are bimodules over the function algebra of the space then morphisms in the representation category are bilinear maps with respect to the action of the function algebra. Internal homomorphisms on the other hand only preserve the right action (in a weak form). Configuration spaces of geometrical notions would be severely restricted by a left linearity condition. This is another strong motivation for modelling geometry on internal homomorphisms.

To illustrate these concepts let us consider the simple example given by the Moyal-Weyl space $\mathbb{R}^{2k}_\Theta$. The noncommutative algebra $A = (C^\infty(\mathbb{R}^{2k}), \star_\Theta)$ corresponding to $\mathbb{R}^{2k}_\Theta$ can be considered as an algebra object in the representation category of the universal enveloping algebra H of the $2k$-dimensional Abelian Lie algebra describing infinitesimal translations on $\mathbb{R}^{2k}$. That is there exists an action of the infinitesimal translations on A, which is given by the (Lie) derivative. Vector bundles such as the noncommutative one-forms and vector fields on $\mathbb{R}^{2k}_\Theta$ are A-bimodules which are equipped with an action of the infinitesimal translations in terms of the Lie derivative, and thereby become objects in the category H–Bimod(A) of H-module A-bimodules. In physical applications one studies geometric structures on $\mathbb{R}^{2k}_\Theta$, which are maps g between such H-module A-bimodules. At this point it differs drastically if we regard g as a morphism in H–Bimod(A) or as an internal homomorphism. In the first case the map g has to be compatible with the left and right A-actions as well as the left H-action describing infinitesimal translations. If we express g as a module map (i.e. morphism in the category), its components $g_{\mu\nu} \in A$ have to be constant as a consequence of translation invariance. A finer consideration of the example above where bimodule connections are modelled as module maps reveals that the space of field configurations of the connections in this case is very small because of the requirement left A-linearity. Therefore, describing geometric structures by morphisms in the category H–Bimod(A) leads to a very rigid framework which does not permit dynamical fields on $\mathbb{R}^{2k}_\Theta$. On the other hand, if we allow g to be an internal homomorphism, which in the present case means that g is a right A-linear map which is not necessarily compatible with the left A-action and the left H-action, the components $g_{\mu\nu}$ are much less constrained, leading to a richer framework for describing noncommutative geometries on $\mathbb{R}^{2k}_\Theta$.

Because we are also interested in *nonassociative* generalisations of noncommutative geometry, in this thesis we extend the constructions developed in [6] to the context of the representation category of a triangular *quasi*-Hopf algebra.

1.2 Overview and Outline

1.2.1 Overview

In this thesis we develop the theory of internal module homomorphisms and connections on bimodules, together with their tensor product structure, for a large class of noncommutative and nonassociative spaces. For this, we take an approach based on category theory. The language of category theory systematically highlights the general structures involved in a model-independent way. An analogous approach was taken in [58] to develop the applications of nonassociative algebras to non-geometric string theory which were discussed in [57]. However, their categories are completely different from ours, and moreover their algebras have the physically undesirable feature that the classical limit only coincides with the algebra of functions on a manifold up to Morita equivalence; instead, the constructions in this thesis always reduce exactly to the classical algebras of functions. We also consider physical applications to noncommutative and nonassociative Yang-Mills theory and Einstein-Cartan gravity, as first steps towards more elaborate models relevant to non-geometric flux deformations of geometry in closed string theory.

From a more technical point of view, we consider the representation category $[H, \mathscr{M}]$ of a quasitriangular quasi-Hopf algebra H and develop some elements of differential geometry *internal* to this category. It is well known that the representation category of a quasi-Hopf algebra is a (weak) monoidal category, which for quasitriangular quasi-Hopf algebras carries the additional structure of a braided monoidal category. We consider algebra objects in the category $[H, \mathscr{M}]$, which due to the generally non-trivial associator are nonassociative algebras of the type found in non-geometric string theory (i.e. they are weakly associative). We also make use of the braiding determined by the quasitriangular structure on the quasi-Hopf algebra H and consider the algebra object A in $[H, \mathscr{M}]$ to be braided commutative (i.e. the product is preserved by the braiding in $[H, \mathscr{M}]$). Given any commutative algebra object A in $[H, \mathscr{M}]$ we then consider symmetric A-bimodule objects in $[H, \mathscr{M}]$ (the symmetry condition being that the left and right A-module structures are identified by the braiding), the collection of which forms a braided monoidal category $H\text{–Bimod}(A)^{\mathrm{sym}}$.

In the spirit of noncommutative geometry the monoidal category $H\text{–Bimod}(A)^{\mathrm{sym}}$ can be geometrically interpreted as the category of all noncommutative and nonassociative H-equivariant vector bundles over the noncommutative and nonassociative space A. The morphisms in this category preserve both the H-module and A-bimodule structures. In contrast to earlier categorical approaches to nonassociative geometry pursued in [58, 13], in which geometric quantities such as (Riemannian) metrics and curvatures are described using morphisms, we describe geometric quantities using the larger class of *internal homomorphisms* of the monoidal category $H\text{–Bimod}(A)^{\mathrm{sym}}$. As motivated in Subsection 1.1.3 many geometric quantities are *not necessarily* H-invariant and hence they cannot be identified with morphisms in $H\text{–Bimod}(A)^{\mathrm{sym}}$. In particular, in situations where the geometric quantities are dynamical (e.g. the metric field in gravity or the curvature field of a connection in Yang-Mills theory) the internal homomorphism point of view is indispensable. We give an explicit description of the internal hom-functor on $H\text{–Bimod}(A)^{\mathrm{sym}}$ in terms of the internal hom-functor on the category $[H, \mathscr{M}]$ and an equalizer which formalizes a weak "right A-linearity condition". This internal homomorphism point of view is inspired by the formalism of [6] and it clarifies and generalises the constructions in [6] and [45]. For internal homomorphisms in a closed braided monoidal category there are evaluation, composition and tensor product morphisms which we explicitly describe in detail. These are the appropriate structures with which to use internal homomorphisms correctly as map-like objects in $[H, \mathscr{M}]$. Although in the category $[H, \mathscr{M}]$ internal homomorphisms are k-linear maps they do not give the correct behaviour under the usual structures. Physically the existence of a tensor product operation for internal homomorphisms means that there is a tensor product operation for constructing noncommutative and nonassociative tensor fields, which is an indispensable tool for describing physical theories such as gravity and other field theories for spaces such as our motivating example of the R-flux compactification.

Promoting the category $[H, \mathscr{M}]$ to a category of bounded $\mathbb{Z}$-graded H-modules we develop the notion of differential calculus (A, d) in $[H, \mathscr{M}]$. We then study connections on objects in H–Bimod$(A)^{\mathrm{sym}}$ from an internal point of view. We formulate the notion of connection on a commutative and associative bimodule in H–Bimod$(A)^{\mathrm{sym}}$ by using the internal hom-functor for the closed monoidal category $[H, \mathscr{M}]$ together with an equalizer which formalizes a suitable generalization of the graded Leibniz rule that is consistent with the structures in $[H, \mathscr{M}]$. The class of connections this technique produces is much larger than that found by the technique used in [13] in which connections are assumed to be bimodule connections equivariant with respect to the H-action. We also develop appropriate morphisms to lift connections in $[H, \mathscr{M}]$ to tensor products and internal hom-objects in the closed braided monoidal category H–Bimod$(A)^{\mathrm{sym}}$. It is important to notice that our notion of tensor product connections differs from the standard one: Although our techniques are only applicable to braided commutative algebras and their bimodules in $[H, \mathscr{M}]$, they are more flexible in the sense that *any* two connections can be lifted to a tensor product connection, not only those which satisfy the very restrictive 'bimodule connection' property proposed in [54, 33, 42, 32]. We also develop a lifting prescription for connections to internal homomorphisms $\hom_A(V, W)$ of objects V, W in H–Bimod$(A)^{\mathrm{sym}}$. These lifts are all important ingredients in (noncommutative and nonassociative) Riemannian geometry for extending e.g. tangent bundle connections to all tensor fields, and they play an instrumental role in physical applications of our formalism to noncommutative and nonassociative gravity theories such as those anticipated to arise in non-geometric string theory. All of these constructions moreover generalize and clarify the corresponding constructions of [6] in categorical terms.

Throughout we systematically study how each structure deforms under cochain twisting. This allows us to obtain a large class of examples of noncommutative and nonassociative geometries by cochain twisting the example of classical differential geometry. In this case, by fixing any Lie group G and any G-manifold M, there is the braided monoidal category of G-equivariant vector bundles over M. We construct a braided monoidal functor from this category to the category H–Bimod$(A)^{\mathrm{sym}}$, where

$A = C^\infty(M)$ is the algebra of functions on M and $H = U\mathfrak{g}$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G (with trivial R-matrix). Then choosing *any* cochain twist based on H we twist the braided monoidal category H–Bimod$(A)^{\mathrm{sym}}$ into a braided monoidal category which describes noncommutative and nonassociative vector bundles over a noncommutative and nonassociative space.

We show that cochain twisting can be understood as a categorical equivalence of closed braided monoidal categories preserving all limits and colimits (and in particular equalisers) between the undeformed and deformed categories. This equivalence then includes the internal homomorphisms, which in our physical interpretation implies that the configuration spaces of deformed geometric quantities are in bijective correspondence with the undeformed ones. This solves the problem of quantum rigidity encountered in the usual approach using bimodule connections. We note that this equivalence is purely on the structural level. Since the deformed Lagrangians should be constructed out of $\star$-products (cf. Chapter 4), while the undeformed ones out of ordinary products, the selection criteria for which physical quantities are realized in nature (e.g. as a critical point of an action) will differ in the deformed and the undeformed case.

We conclude by unpacking and making explicit the somewhat abstract categorical constructions of our formalism in a less formal language. We focus on the special case of most physical relevance: the cochain twist quantization of a classical manifold. The formalism is powerful enough to capture the cases of constant non-geometric fluxes as well as non-constant ones such as those which arise in the flux formulation of double field theory [18]; in fact, our constructions are completely general and can be applied to a much broader framework without specific reference to string theory. We further restrict to trivial vector bundles over these noncommutative and nonassociative spaces with an action of the pertinent Hopf algebra of symmetries of the non-geometric background. This simplification enables us to give very explicit "local" descriptions of the noncommutative and nonassociative geometry while still retaining generic features and indicating how the general formalism developed in the rest of the thesis may be applied to constructions of physically viable field theories. As a starting point for building more elaborate models describing the low-energy effective dynamics of closed strings in non-geometric backgrounds, we demonstrate how to apply our formalism to the constructions of physically sensible action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces; our considerations are based on Einstein-Cartan geometry and its noncommutative generalization which was developed in [3].

1.2.2 Outline

Let us now give a brief outline of the contents of this thesis. Chapter 2 is a technical chapter based on [34]. Chapter 3 contains the core contribution of this thesis and is based on [37]. Chapter 4 contains examples and concrete realisations of the formalism developed in Chapters 2 and 3 and is based on [35]. Definitions in category theory of specific relevance to the constructions of this thesis are collected in Appendix A and Appendix B contains additional calculations. We conclude with a brief summary and outlook in Chapter 5.

In Chapter 2 we systematically study noncommutative and nonassociative algebras A and their bimodules as algebras and bimodules internal to the representation category of a quasitriangular quasi-Hopf algebra. We enlarge the morphisms of the monoidal category of A-bimodules by internal homomorphisms, and describe explicitly their evaluation, composition and tensor product morphisms. We show that for braided commutative algebras A the subcategory of symmetric A-bimodule objects is also a braided closed monoidal category. We systematically describe how these structures deform under cochain twisting of the quasi-Hopf algebra. These constructions set up the basic ingredients for the development of differential geometry internal to a quasi-Hopf representation category and applications to models of noncommutative and nonassociative gravity such as those anticipated from non-geometric string theory.

Throughout we make all of our constructions explicit, even when they follow easily from abstract arguments of category theory, in order to set up a concrete computational framework for Chapter 4. In particular, in contrast to what is sometimes done in the literature, we pay careful attention to associator insertions: Although by the coherence theorems there is no loss of generality in imposing the strictness property on a monoidal category (i.e. strong associativity of the monoidal structure), for our computational purposes we are careful not to mix up equality and isomorphy of objects.

In Section 2.1 we describe the closed symmetric monoidal category $\mathscr{M}$ of k-modules (for an arbitrary ring k) which is the category on which the constructions in Chapter 2 are based. We also discuss the subcategory of bimodules over an algebra object in $\mathscr{M}$. We end the section by recalling the definition of a quasi-Hopf algebra H and of cochain twisting of quasi-Hopf algebras.

In Section 2.2 we recall the definition of the monoidal category of (left) H-modules $[H, \mathscr{M}]$ over k. By explicitly constructing an internal hom-functor for this category, we show that $[H, \mathscr{M}]$ is also a closed monoidal category, and we describe explicitly the canonical evaluation and composition morphisms for the internal hom-objects. By restricting to quasi-Hopf algebras H which are quasitriangular, we endow the representation category $[H, \mathscr{M}]$ with the additional structure of a braiding with which we define commutative algebra objects in $[H, \mathscr{M}]$ and explicitly describe the canonical tensor product morphisms for the internal hom-objects. We also define an internal commutator which endows the algebra of internal endomorphisms on an object with the structure of a Lie algebra and show how the morphisms in $[H, \mathscr{M}]$ are embedded in the internal homomorphisms in $[H, \mathscr{M}]$.

In Section 2.3 we introduce symmetric bimodules over commutative algebra objects A in the category $[H, \mathscr{M}]$. We show that the category of symmetric A-bimodules in $[H, \mathscr{M}]$ also forms a braided closed monoidal category H–Bimod$(A)^{\mathrm{sym}}$: We explicitly construct a monoidal functor and internal hom-functor for the category H–Bimod$(A)^{\mathrm{sym}}$ and show that the braiding in $[H, \mathscr{M}]$ descends to a braiding in H–Bimod$(A)^{\mathrm{sym}}$. We show also that cochain twisting by a cochain twisting element $F \in H \otimes H$ leads to an equivalence between the monoidal categories H–Bimod$(A)^{\mathrm{sym}}$ and H_F–Bimod$(A_F)^{\mathrm{sym}}$ with the deformed algebra A_F in $[H_F, \mathscr{M}]$ functorially assigned to the algebra A in $[H, \mathscr{M}]$.

In Chapter 3 we promote the category $\mathscr{M}$ to a category of bounded $\mathbb{Z}$-graded k-modules which we denote by the same symbol $\mathscr{M}$. The algebra objects in Chapter 2 then lie in degree 0 of the commutative algebra objects in the graded category $\mathscr{M}$. Bimodules over these algebra objects are symmetric bimodules in$\mathscr{M}$. We systematically proceed to formulate notions of classical differential geometry internal to the representation category $[H, \mathscr{M}]$ of an arbitrary triangular quasi-Hopf algebra H. We describe differential calculi and connections using universal categorical constructions to capture algebraic properties such as Leibniz rules. Our main result is the construction of morphisms which provide prescriptions for lifting connections to tensor products and to internal homomorphisms. We also describe the curvatures of connections within this formalism.

We begin in Section 3.1 with a brief review of the categorical framework which was developed in Chapter 2 but now in the context of $\mathbb{Z}$-graded modules; this allows us later on to regard graded objects such as differential calculi naturally as objects in the category.

In Section 3.2 we introduce derivations $\mathrm{der}(A)$ on braided commutative algebras A in $[H, \mathscr{M}]$ by formalizing the Leibniz rule in terms of an equalizer in $[H, \mathscr{M}]$. We analyse structural properties of $\mathrm{der}(A)$ and in particular prove that, in the case where H is triangular, $\mathrm{der}(A)$ together with an internal commutator $[\,\cdot\,,\,\cdot\,]$ is a Lie algebra in $[H, \mathscr{M}]$. We then introduce differential operators $\mathrm{diff}(V)$ on symmetric A-bimodules V in $[H, \mathscr{M}]$ by again using a suitable equalizer in $[H, \mathscr{M}]$ to capture the relevant algebraic properties. We show that $\mathrm{diff}(V)$ is an algebra in $[H, \mathscr{M}]$ and we also prove that the zeroth order differential operators are the internal endomorphisms $\mathrm{end}_A(V)$ in the category of symmetric A-bimodules H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. Using the product structure on differential operators to formalize nilpotency of a differential, we then give a definition of a differential calculus in $[H, \mathscr{M}]$.

In Section 3.3 we develop a notion of connections $\mathrm{con}(V)$ on objects V in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. The idea is to formalize a generalization of the usual Leibniz rule with respect to a differential calculus in terms of an equalizer in $[H, \mathscr{M}]$. The resulting object $\mathrm{con}(V)$ is analysed in detail and it is shown that the usual affine space of ordinary connections arises as a certain proper subset of $\mathrm{con}(V)$. Our more flexible definition of connections has the advantage that $\mathrm{con}(V)$ also forms an object in $[H, \mathscr{M}]$ in addition to being an affine space. We then develop a lifting prescription for connections to tensor products $V \otimes_A W$ of objects V, W in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. We also develop a lifting prescription for connections to internal homomorphisms $\mathrm{hom}_A(V, W)$ of objects V, W in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ and show that cochain twist quantisation preserves structurally these constructions by the same isomorphism which preserves the internal endomorphism objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Finally, in Section 3.4 we assign curvatures to connections and show that they are internal endomorphisms in the category H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$, provided that H is triangular. We also obtain a Bianchi tensor, which in classical differential geometry would identically vanish; in general it is not necessarily equal to 0, and hence in this sense it characterises the noncommutativity and nonassociativity of our geometries. We further observe that the curvature of any tensor product connection is the sum of the two individual curvatures, which means that curvatures behave additively in an appropriate sense.

In Chapter 4 we apply the constructions in Chapter 2 to the concrete examples of deformation quantization of G-equivariant vector bundles over G-manifolds where G is the Lie group obtained by exponentiating the Lie algebra of (a subset of the) infinitesimal diffeomorphisms of a manifold, and provide examples of noncommutative and nonassociative spaces which fit into this framework which include the Q and R-flux compactifications of closed string theory. We also consider how the constructions in Chapter 3 may be applied in the simplest model of cochain twist deformations of *trivial* vector bundles over noncommutative and nonassociative spaces and provide physically viable action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces, as first steps towards more elaborate models relevant to non-geometric flux deformations of geometry in closed string theory.

In Section 4.1 we construct concrete examples for the categories H–$\mathrm{Alg}^{\mathrm{com}}$ and H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ for a given braided commutative algebra $A \in H$–$\mathrm{Alg}^{\mathrm{com}}$ starting from ordinary differential geometry. In these examples the algebras A and bimodules V are commutative, i.e. braided commutative with respect to the trivial R-matrix $R = 1 \otimes 1$. Deformation quantization by cochain twists then leads to examples of noncommutative and also nonassociative algebras and bimodules.

In Sections 4.2 we restrict to trivial vector bundles over noncommutative and nonassociative spaces with diagonal action of the pertinent Hopf algebra of symmetries of the non-geometric background. We give concrete realizations of the pertinent bimodule operations for homomorphism bundles. In Section 4.3 we apply this framework to obtain explicit expressions for connections and their curvatures on noncommutative and nonassociative vector bundles and in Section 4.4 we demonstrate how to apply our formalism to the constructions of physically sensible action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces.

Chapter 2

Mathematical foundations

This chapter sets up the basic framework for the formalism of noncommutative and nonassociative differential geometry to be developed in Chapter 3. In Section 2.1 we review the theory of closed braided monoidal categories for the category of k-modules and the theory of cochain twisting of quasitriangular quasi-Hopf algebras. In Sections 2.2 and 2.3 we interpret the theory of Section 2.1 in the monoidal category $[H, \mathscr{M}]$ for an arbitrary quasitriangular quasi-Hopf algebra H showing how all structures are preserved by cochain twisting. We also show the important result for physics that the morphisms in $[H, \mathscr{M}]$ are contained in the internal homomorphisms in a structure preserving way.

2.1 Preliminaries

2.1.1 k-modules

Throughout this Chapter k denotes a commutative and associative ring with unit $1 \in k$. In examples $k = \mathbb{K}[\hbar]$ where $\hbar$ is a formal deformation parameter and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

The constructions in this Chapter are based on the category of k-modules $\mathscr{M} := \mathrm{Mod}_k$. The objects in $\mathscr{M}$ are k-modules and the morphisms are k-linear maps.

The category $\mathscr{M}$ is (strict) monoidal with monoidal functor given by the tensor product of k-modules simply denoted by $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ without any subscript. To any $\mathscr{M} \times \mathscr{M}$-morphism $(f : V \to V',\ g : W \to W')$ the monoidal functor assigns the k-linear map

$$f \otimes g : V \otimes W \longrightarrow V' \otimes W' \ , \qquad v \otimes w \longmapsto f(v) \otimes g(w) \ . \tag{2.1.1}$$

The unit object in $\mathscr{M}$ is the one-dimensional k-module k. The associator in $\mathscr{M}$ is the natural isomorphism

$$\Phi : \otimes \circ \big(\otimes \times \mathrm{id}_{\mathscr{M}} \big) \Longrightarrow \otimes \circ \big(\mathrm{id}_{\mathscr{M}} \times \otimes \big) \ , \tag{2.1.2}$$

given by the identity maps. For the rest of this section the associator will be trivial. The unitors in $\mathscr{M}$ are the natural isomorphisms

$$\lambda : k \otimes - \Longrightarrow \mathrm{id}_{\mathscr{M}} \qquad \text{and} \qquad \varrho : - \otimes k \Longrightarrow \mathrm{id}_{\mathscr{M}} \ , \tag{2.1.3}$$

where $\mathrm{id}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is the identity functor and $k \otimes - : \mathscr{M} \to \mathscr{M}$ is the functor assigning to an object V in $\mathscr{M}$ the k-module $k \otimes V$ and to an $\mathscr{M}$-morphism $f : V \to W$ the k-linear map $\mathrm{id}_k \otimes f : k \otimes V \to k \otimes W \ , \ c \otimes v \mapsto c \otimes f(v)$. The functor $- \otimes k : \mathscr{M} \to \mathscr{M}$ is defined similarly. The V-components of the unitors are given by $\lambda_V : k \otimes V \to V \ , \ c \otimes v \mapsto c\, v$ and $\varrho_V : V \otimes k \to V \ , \ v \otimes c \mapsto c\, v$.

The monoidal category $\mathscr{M}$ of k-modules admits an internal hom-structure which we shall describe below. This internal hom-structure plays a central role in the rest of this thesis. Internal homomorphisms are similar to morphisms in a category, but whereas morphisms preserve every structure on the objects of the category, internal homomorphisms are not subject to this strict requirement.

Proposition 2.1.1 (Hom functor)**.** *The assignment of Hom-sets in a locally small category $\mathscr{C}$ is functorial: The* Hom-functor *is the functor*

$$\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Set} \ , \ (V, W) \mapsto \mathrm{Hom}_{\mathscr{C}}(V, W) \ ,$$
$$(f^{\mathrm{op}}, g) \mapsto g \circ - \circ f \ . \tag{2.1.4}$$

Proof. $\mathrm{Hom}_{\mathscr{C}}$ clearly preserves the identity morphisms $\mathrm{hom}(\mathrm{id}_V^{\mathrm{op}}, \mathrm{id}_W) = \mathrm{id}_{\mathrm{hom}(V,W)}$. It also preserves compositions

$$\mathrm{hom}(f^{\mathrm{op}} \circ^{\mathrm{op}} \tilde{f}^{\mathrm{op}}, g \circ \tilde{g})(\,\cdot\,) = g \circ \tilde{g} \circ (\,\cdot\,) \circ \tilde{f} \circ f = \big(\mathrm{hom}(f^{\mathrm{op}}, g) \circ \mathrm{hom}(\tilde{f}^{\mathrm{op}}, \tilde{g})\big)(\,\cdot\,) \ , \tag{2.1.5}$$

for any two composable morphisms (f^{op}, g) and $(\tilde{f}^{\mathrm{op}}, \tilde{g})$ in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. $\square$

Definition 2.1.2 (Representable functor)**.** Let $\mathscr{C}$ be a locally small category. An object $X \in \mathscr{C}$ *represents* the functor $G : \mathscr{C} \to \mathsf{Set}$ if

$$G \cong \mathrm{Hom}_{\mathscr{C}}(-,\ X)\ , \tag{2.1.6}$$

are equivalent as functors (cf. Definition A.2.5). G is then said to be representable.

Definition 2.1.3 (Internal homomorphism)**.** Given a (locally small) monoidal category $\mathscr{C}$ and any two objects $V, W \in \mathscr{C}$, an internal homomorphism object $\mathrm{hom}(V, W)$ in $\mathscr{C}$ is an object in $\mathscr{C}$ which represents the functor $\mathrm{Hom}_{\mathscr{C}}\big(- \otimes V, W\big) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$.

Definition 2.1.4 (Closed monoidal category)**.** A *closed monoidal category* is a monoidal category $\mathscr{C}$ which permits internal hom-objects: For any two objects $V, W \in \mathscr{C}$, there is an object $\mathrm{hom}(V, W)$ which represents the functor $\mathrm{Hom}_{\mathscr{C}}\big(- \otimes V, W\big) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$. The natural bijection

$$\zeta_{-,V,W} : \mathrm{Hom}_{\mathscr{C}}\big(- \otimes V, W\big) \longrightarrow \mathrm{Hom}_{\mathscr{C}}\big(-, \mathrm{hom}(V, W)\big)\ . \tag{2.1.7}$$

is traditionally referred to as the *currying bijection*. From this equation and Definition 2.1.1 it is clear that the internal hom-objects assign objects in $\mathscr{C}$ to objects in $\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}$. The assignment of internal hom-objects is functorial and we denote the corresponding functor by

$$\mathrm{hom} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{C}\ . \tag{2.1.8}$$

(We shall use a subscript later on to distinguish internal hom-functors in different categories.)

$\mathscr{M}$ is a closed monoidal category with internal hom-functor

$$\mathrm{hom} : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathscr{M}\ , \tag{2.1.9}$$

which assigns to any object (V,W) in $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$, the object

$$\hom\big(V,W\big) := \mathrm{Hom}_{\mathscr{M}}\big(V,W\big) \ , \tag{2.1.10}$$

where $\mathrm{Hom}_{\mathscr{M}}(V,W)$ is the k-module of k-linear maps between V and W (in $\mathscr{M}$ there is no distinction between internal homomorphisms and morphisms because $\mathscr{M}$ is enriched over itself). To any $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$-morphism $(f^{\mathrm{op}} : V \to V', g : W \to W')$ the internal hom-functor assigns the $\mathscr{M}$-morphism

$$\hom(f^{\mathrm{op}}, g) : \hom\big(V,W\big) \longrightarrow \hom\big(V',W'\big) \ , \qquad L \longmapsto g \circ L \circ f \ . \tag{2.1.11}$$

Functoriality of hom follows from that of $\mathrm{Hom}_{\mathscr{M}}$: $\mathrm{Hom}_{\mathscr{M}}$ clearly preserves the identity morphisms $\mathrm{Hom}_{\mathscr{M}}(\mathrm{id}_V^{\mathrm{op}}, \mathrm{id}_W) = \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)}$. It also preserves compositions

$$\begin{aligned}\mathrm{Hom}_{\mathscr{M}}(f^{\mathrm{op}} \circ^{\mathrm{op}} \tilde{f}^{\mathrm{op}}, g \circ \tilde{g})(\,\cdot\,) &= g \circ \tilde{g} \circ (\,\cdot\,) \circ \tilde{f} \circ f \\ &= \big(\mathrm{Hom}_{\mathscr{M}}(f^{\mathrm{op}}, g) \circ \mathrm{Hom}_{\mathscr{M}}(\tilde{f}^{\mathrm{op}}, \tilde{g})\big)(\,\cdot\,) \ ,\end{aligned} \tag{2.1.12}$$

for any two composable morphisms (f^{op}, g) and $(\tilde{f}^{\mathrm{op}}, \tilde{g})$ in $\mathscr{M}$. The natural currying bijection

$$\zeta : \mathrm{Hom}_{\mathscr{M}}(- \otimes -, -) \Longrightarrow \mathrm{Hom}_{\mathscr{M}}(-, \hom(-,-)) \tag{2.1.13}$$

has (V,W,X)-component

$$\zeta_{V,W,X}(f) : V \longrightarrow \hom(W,X) \ , \qquad v \longmapsto f\big(v \otimes (\,\cdot\,)\big) \ , \tag{2.1.14}$$

for all $\mathscr{M}$-morphisms $f : V \otimes W \to X$, and the inverse currying bijection has (V,W,X)-component

$$\zeta^{-1}_{V,W,X}(g) : V \otimes W \longrightarrow X \ , \qquad v \otimes w \longmapsto g(v)(w) \ , \tag{2.1.15}$$

for all $\mathscr{M}$-morphisms $g : V \to \hom(W,X)$. A straightforward calculation shows that $\zeta^{-1}_{V,W,X}$ is indeed the inverse of the map $\zeta_{V,W,X}$. To confirm that $\zeta_{V,W,X}$ is the (V,W,X)-component of a natural isomorphism ζ between the two functors $\mathrm{Hom}_{\mathscr{M}}(-\otimes -,-)$ and $\mathrm{Hom}_{\mathscr{M}}(-,\mathrm{hom}(-,-))$ from $\big(\mathscr{M}\big)^{\mathrm{op}}\times\big(\mathscr{M}\big)^{\mathrm{op}}\times\mathscr{M}$ to the category of sets, take any morphism $\big(f_V^{\mathrm{op}}:V\to V', f_W^{\mathrm{op}}:W\to W', f_X:X\to X'\big)$ in $\big(\mathscr{M}\big)^{\mathrm{op}}\times\big(\mathscr{M}\big)^{\mathrm{op}}\times\mathscr{M}$; one has that the diagram (in the category Sets)

$$
\begin{tikzcd}
\mathrm{Hom}\big(V\otimes W,X\big) \arrow[r, "\zeta_{V,W,X}"] \arrow[d, "\mathrm{Hom}(f_V^{\mathrm{op}}\otimes f_W^{\mathrm{op}},f_X)"'] & \mathrm{Hom}\big(V,\mathrm{hom}(W,X)\big) \arrow[d, "\mathrm{Hom}(f_V^{\mathrm{op}},\mathrm{hom}(f_W^{\mathrm{op}},f_X))"] \\
\mathrm{Hom}\big(V'\otimes W',X'\big) \arrow[r, "\zeta_{V',W',X'}"'] & \mathrm{Hom}\big(V',\mathrm{hom}(W',X')\big)
\end{tikzcd}
\tag{2.1.16}
$$

commutes. Indeed, for any $\mathscr{M}$-morphism $f:V\otimes W\to X$

$$
\begin{aligned}
&\mathrm{Hom}(f_V^{\mathrm{op}},\mathrm{hom}(f_W^{\mathrm{op}},f_X))\big(\zeta_{V,W,X}(f)\big)\\
&\quad= f_X\circ f\Big(f_V(\,\cdot\,)\otimes(\,\cdot\,)\Big)\circ f_W\\
&\quad= f_X\circ f\Big(f_V\otimes f_W\big((\,\cdot\,)\otimes(\,\cdot\,)\big)\Big)\\
&\quad= \zeta_{V',W',X'}\big(\mathrm{Hom}(f_V^{\mathrm{op}}\otimes f_W^{\mathrm{op}},f_X)(f)\big)\ .
\end{aligned}
\tag{2.1.17}
$$

For any closed monoidal category $\mathscr{C}$ there exist canonical evaluation and composition morphisms for the internal hom-objects [52, Proposition 9.3.13]. These morphisms are derived using the currying bijection, see e.g. [52, Proposition 9.3.13], and they induce important structures on the internal homomorphisms which give them map-like properties compatible with the structure of an object in the category.

Proposition 2.1.5. *Let $\mathscr{C}$ be any (locally small) closed monoidal category with internal hom-functor* $\mathrm{hom}:\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{C}$. *Then there are $\mathscr{C}$-morphisms*

$$\mathrm{ev}_{V,W}\ :\ \mathrm{hom}(V,W)\otimes V\longrightarrow W\ , \tag{2.1.18a}$$

$$\bullet_{V,W,X}\ :\ \mathrm{hom}(W,X)\otimes\mathrm{hom}(V,W)\longrightarrow\mathrm{hom}(V,X)\ , \tag{2.1.18b}$$

for all objects V,W,X in $\mathscr{C}$.

Proof. To construct the $\mathscr{C}$-morphism $\mathrm{ev}_{V,W}$ let us notice that, due to the currying, there is a bijection of Hom-sets

$$\mathrm{Hom}_{\mathscr{C}}\big(\hom(V,W),\hom(V,W)\big) \xrightarrow{\zeta^{-1}_{\hom(V,W),V,W}} \mathrm{Hom}_{\mathscr{C}}\big(\hom(V,W)\otimes V,W\big) \ , \tag{2.1.19}$$

for all objects V,W in $\mathscr{C}$. Choosing the identity $\mathrm{id}_{\hom(V,W)}$ in the Hom-set on the left-hand side, we obtain via this bijection the $\mathscr{C}$-morphism

$$\mathrm{ev}_{V,W} := \zeta^{-1}\big(\mathrm{id}_{\hom(V,W)}\big) : \hom\big(V,W\big)\otimes V \longrightarrow W \ , \tag{2.1.20}$$

for all objects V,W in $\mathscr{C}$. Considering the following composition of $\mathscr{C}$-morphisms

$$\begin{array}{c} \big(\hom(V,W)\otimes\hom(X,V)\big)\otimes X \\ \Big\downarrow{\scriptstyle \Phi_{\hom(V,W),\hom(X,V),X}} \\ \hom(V,W)\otimes\big(\hom(X,V)\otimes X\big) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\hom(V,W)}\otimes\mathrm{ev}_{X,V}} \\ \hom(V,W)\otimes V \\ \Big\downarrow{\scriptstyle \mathrm{ev}_{V,W}} \\ W \end{array} \tag{2.1.21}$$

we define the $\mathscr{C}$-morphism

$$\bullet_{V,W,X} := \zeta\Big(\mathrm{ev}\circ\big(\mathrm{id}_{\hom(V,W)}\otimes\mathrm{ev}\big)\circ\Phi\Big) : \hom\big(V,W\big)\otimes\hom\big(X,V\big) \longrightarrow \hom\big(X,W\big) \ , \tag{2.1.22}$$

for all objects V,W,X in $\mathscr{C}$. $\square$

The evaluation and composition of internal homomorphisms in $\mathscr{M}$ are given by the usual operations: For any three objects V, W, X in $\mathscr{M}$

$$\begin{aligned} \mathrm{ev}_{V,W} \ &: \ \hom(V,W) \otimes V \longrightarrow W \ , \\ & \qquad L \otimes v \mapsto L(v) \ , \end{aligned} \tag{2.1.23a}$$

$$\begin{aligned} \bullet_{V,W,X} \ &: \ \hom(W,X) \otimes \hom(V,W) \longrightarrow \hom(V,X) \ , \\ & \qquad L \otimes K \mapsto L \circ K \ . \end{aligned} \tag{2.1.23b}$$

The properties of the evaluation and composition morphisms correspond to the usual properties for k-linear maps:

$$g(v)(w) = \mathrm{ev}_{W,X} \circ (g \otimes \mathrm{id}_W)(v \otimes w) = \zeta^{-1}_{V,W,X}(g)(v \otimes w) = g(v)(w) \ , \tag{2.1.24a}$$

$$L \circ L'(v) = \mathrm{ev}(L \bullet L' \otimes v) = \mathrm{ev}(L \otimes \mathrm{ev}(L' \otimes v)) = L \circ L'(v) \ , \tag{2.1.24b}$$

$$(L'' \circ L) \circ L' = (L'' \bullet L) \bullet L' = L'' \bullet (L \bullet L') = L'' \circ (L \circ L') \ , \tag{2.1.24c}$$

for any $\mathscr{M}$-morphism $g : V \to \hom(W, X)$, $v \in V$ and $w \in W$, and for internal homomorphisms $L'' \in \hom(X, Y)$, $L \in \hom(W, X)$, $L' \in \hom(V, W)$.

Finally, $\mathscr{M}$ can be equipped with a braiding natural isomorphism

$$\sigma : \otimes \Longrightarrow \otimes^{\mathrm{op}} \ , \tag{2.1.25}$$

(where the opposite tensor product $\otimes^{\mathrm{op}}$ is defined in Definition A.3.6) with (V, W)-component given by

$$\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V \ , \qquad v \otimes w \longmapsto \ w \otimes v \ , \tag{2.1.26}$$

which trivially satisfies the hexagon relations.

Remark 2.1.6. Note that the $\mathscr{M}$-morphism $\sigma_{W,V} \circ \sigma_{V,W} : V \otimes W \to V \otimes W$ coincides with the identity morphism $\mathrm{id}_{V \otimes W}$, hence the braiding is symmetric.

For any closed braided monoidal category $\mathscr{C}$ there exist canonical tensor product morphisms for the internal hom-objects [52, Proposition 9.3.13].

Proposition 2.1.7. *Let $\mathscr{C}$ be any braided monoidal category with internal hom-functor* $\mathrm{hom} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$*. Then there is a $\mathscr{C}$-morphism*

$$\circledbullet_{V,W,X,Y} \,:\, \mathrm{hom}(V,W) \otimes \mathrm{hom}(X,Y) \longrightarrow \mathrm{hom}(V \otimes X, W \otimes Y) \;, \tag{2.1.27}$$

for all objects V, W, X, Y in $\mathscr{C}$.

Proof. Considering the following composition of $\mathscr{C}$-morphisms

$$\begin{array}{c}
\big(\mathrm{hom}(V,W) \otimes \mathrm{hom}(X,Y)\big) \otimes (V \otimes X) \\
\Big\downarrow {\scriptstyle \Phi_{\mathrm{hom}(V,W),\mathrm{hom}(X,Y),V\otimes X}} \\
\mathrm{hom}(V,W) \otimes \big(\mathrm{hom}(X,Y) \otimes (V \otimes X)\big) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\mathrm{hom}(V,W)} \otimes \Phi^{-1}_{\mathrm{hom}(X,Y),V,X}} \\
\mathrm{hom}(V,W) \otimes \big((\mathrm{hom}(X,Y) \otimes V) \otimes X\big) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\mathrm{hom}(V,W)} \otimes (\tau_{\mathrm{hom}(X,Y),V} \otimes \mathrm{id}_X)} \\
\mathrm{hom}(V,W) \otimes \big((V \otimes \mathrm{hom}(X,Y)) \otimes X\big) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\mathrm{hom}(V,W)} \otimes \Phi_{V,\mathrm{hom}(X,Y),X}} \\
\mathrm{hom}(V,W) \otimes \big(V \otimes (\mathrm{hom}(X,Y) \otimes X)\big) \\
\Big\downarrow {\scriptstyle \Phi^{-1}_{\mathrm{hom}(V,W),V,\mathrm{hom}(X,Y)\otimes X}} \\
\big(\mathrm{hom}(V,W) \otimes V\big) \otimes \big(\mathrm{hom}(X,Y) \otimes X\big) \\
\Big\downarrow {\scriptstyle \mathrm{ev}_{V,W} \otimes \mathrm{ev}_{X,Y}} \\
W \otimes Y
\end{array} \tag{2.1.28}$$

we define the $\mathscr{C}$-morphism

$$\begin{aligned}
\circledbullet := \zeta\Big((\mathrm{ev} \otimes \mathrm{ev}) \circ \Phi^{-1} \circ (\mathrm{id} \otimes \Phi) \circ (\mathrm{id} \otimes (\tau \otimes \mathrm{id})) \circ (\mathrm{id} \otimes \Phi^{-1}) \circ \Phi\Big) :& \\
\mathrm{hom}\big(V,W\big) \otimes \mathrm{hom}\big(X,Y\big) \longrightarrow \mathrm{hom}\big(V \otimes X, W \otimes Y\big) \;,&
\end{aligned} \tag{2.1.29}$$

for all objects V, W, X, Y in $\mathscr{C}$. □

The tensor product of internal homomorphisms in $\mathscr{M}$ is given by the usual operation: For any four objects V, W, X, Y in $\mathscr{M}$

$$\circledast_{V,W,X,Y} : \hom(V,W) \otimes \hom(X,Y) \longrightarrow \hom(V \otimes X, W \otimes Y) \ ,$$
$$L \otimes K \mapsto L \otimes K \ . \tag{2.1.30}$$

The compatibility between the composition and tensor product morphism corresponds to the usual properties for k-linear maps: Let V, W, X, Y, Z be any five objects in $\mathscr{M}$. Then

$$(K \bullet L) \circledast (K' \bullet L') = (K \circledast K') \bullet (L \circledast L') \ , \tag{2.1.31}$$

for all $L \in \hom(V,W)$, $K \in \hom(W,X)$, $L' \in \hom(X,Y)$ and $K' \in \hom(Y,Z)$.

The tensor product morphisms satisfy an associativity property which coincides with the usual property for k-linear maps:

$$(L \circledast K) \circledast M = L \circledast (K \circledast M) \ . \tag{2.1.32}$$

In this chapter $\mathscr{M}$ will denote the closed symmetric monoidal category of k-modules equipped with the braiding defined in equation (2.1.26).

2.1.2 Algebras in $\mathscr{M}$

An algebra in $\mathscr{M}$ is a monoid object (A, μ_A, η_A) in the (monoidal) category $\mathscr{M}$. In other words

Definition 2.1.8 (Algebra)**.** An *algebra* in $\mathscr{M}$ is an object A in $\mathscr{M}$ together with two $\mathscr{M}$-morphisms $\mu_A : A \otimes A \to A$ (product) and $\eta_A : k \to A$ (unit) such that the diagrams

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\mu_A\otimes \mathrm{id}_A"] \arrow[d, "\Phi_{A,A,A}"'] & & A\otimes A \arrow[dd, "\mu_A"] \\
A\otimes (A\otimes A) \arrow[d, "\mathrm{id}_A\otimes\mu_A"'] & & \\
A\otimes A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd} \tag{2.1.33a}$$

$$\begin{tikzcd}
k\otimes A \arrow[d, "\eta_A\otimes \mathrm{id}_A"'] \arrow[dr, "\lambda_A"] & \\
A\otimes A \arrow[r, "\mu_A"'] & A
\end{tikzcd} \qquad
\begin{tikzcd}
& A\otimes k \arrow[d, "\mathrm{id}_A\otimes\eta_A"] \arrow[dl, "\varrho_A"'] \\
A & A\otimes A \arrow[l, "\mu_A"]
\end{tikzcd} \tag{2.1.33b}$$

in $\mathscr{M}$ commute. We shall denote by Alg the category with objects all algebras in $\mathscr{M}$ and morphisms given by all structure preserving $\mathscr{M}$-morphisms, i.e. an Alg-morphism $f : A \to B$ is an $\mathscr{M}$-morphism such that $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

Given an algebra A in $\mathscr{M}$ it is convenient to use a short-hand notation to denote the product of elements by $\mu_A(a \otimes a') = a\,a'$, for all $a, a' \in A$. In this short-hand notation, since the associator in $\mathscr{M}$ is trivial, the first diagram in Definition 2.1.8 implies that

$$(a\,a')\,a'' = a\,(a'\,a'') \ , \tag{2.1.34}$$

for all $a, a', a'' \in A$, and denoting the unit element in A by $1_A := \eta_A(1)$, the last two diagrams in Definition 2.1.8 imply that

$$1_A\,a = a = a\,1_A \ , \tag{2.1.35}$$

for all $a \in A$. Then an Alg-morphism $f : A \to B$ is a k-linear map that satisfies

$$f(a\,a') = f(a)\,f(a') \quad , \quad f(1_A) = 1_B \quad , \tag{2.1.36}$$

for all $a, a' \in A$.

Example 2.1.9. Given any object V in $\mathscr{M}$ we can consider its internal endomorphisms $\mathrm{end}(V) := \mathrm{hom}(V, V)$, which is an object in $\mathscr{M}$. By Proposition 2.1.5 there is an $\mathscr{M}$-morphism

$$\mu_{\mathrm{end}(V)} := \bullet_{V,V,V} : \mathrm{end}(V) \otimes \mathrm{end}(V) \longrightarrow \mathrm{end}(V) \ . \tag{2.1.37}$$

Explicitly, the composition morphism is given in (2.1.23). Furthermore, due to the currying ζ in (2.1.14) we can assign to the $\mathscr{M}$-morphism $\lambda_V : k \otimes V \to V$ the $\mathscr{M}$-morphism

$$\eta_{\mathrm{end}(V)} := \zeta_{k,V,V}(\lambda_V) : k \longrightarrow \mathrm{end}(V) \ . \tag{2.1.38}$$

Explicitly, evaluating this morphism on $1 \in k$ we find $1_{\mathrm{end}(V)} := \eta_{\mathrm{end}(V)}(1) = 1_{\mathrm{end}_k(V)} \in \mathrm{end}(V)$. Using (2.1.24) it is clear that $\big(\mathrm{end}(V), \mu_{\mathrm{end}(V)}, \eta_{\mathrm{end}(V)}\big)$ satisfies the axioms for an algebra in $\mathscr{M}$.

Remark 2.1.10. For any object V in $\mathscr{M}$ the algebra $\mathrm{end}(V)$ in $\mathscr{M}$ describes the algebra of linear operators on V. A *representation* of an object A in Alg on V in $\mathscr{M}$ is then defined to be an Alg-morphism $\pi_A : A \to \mathrm{end}(V)$.

There is an internal commutator for the internal endomorphisms (to simplify notation we drop indices on morphisms in this definition and its consequences):

Definition 2.1.11 (Internal commutator)**.** Let V be an object in $\mathscr{M}$. The *internal commutator* in the algebra of internal endomorphisms $\mathrm{end}(V)$ is the $\mathscr{M}$-morphism $[\,\cdot\,,\,\cdot\,] : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$ defined by

$$[\,\cdot\,,\,\cdot\,] := \bullet - \bullet \circ \sigma \ .$$

That is $[L, L'] = L \circ L' - L' \circ L$ for all $L, L' \in \mathrm{end}(V)$.

Remark 2.1.12. Notice that the target of a morphism is an object, so the commutator of internal endomorphisms is indeed an internal endomorphism.

Proposition 2.1.13. *Let V be an object in $\mathscr{M}$. The internal commutator in* $\mathrm{end}(V)$ *satisfies the following properties:*

(i) $[\,\cdot\,,\,\cdot\,]$ *is braided antisymmetric*

$$[\,\cdot\,,\,\cdot\,] = -[\,\cdot\,,\,\cdot\,] \circ \sigma \ , \tag{2.1.39}$$

i.e. $\big[L, L'\big] = -\big[L', L\big]$ *for all* $L, L' \in \mathrm{end}(V)$.

(ii) $[\,\cdot\,,\,\cdot\,]$ *satisfies the braided Jacobi identity* $\mathrm{Jac} = 0$*, with Jacobiator given by the* $\mathscr{M}$*-morphism* $\mathrm{Jac} : (\mathrm{end}(V) \otimes \mathrm{end}(V)) \otimes \mathrm{end}(V) \longrightarrow \mathrm{end}(V)$ *defined as*

$$\mathrm{Jac} := [\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \big(((\mathrm{id} \otimes \mathrm{id}) \otimes \mathrm{id}) + (\sigma \circ \Phi) + (\Phi^{-1} \circ \sigma)\big) \ , \tag{2.1.40}$$

i.e. $\big[\big[L, L'\big], L''\big] + \big[\big[L', L''\big], L\big] + \big[\big[L'', L\big], L'\big]$ *for all* $L, L', L'' \in \mathrm{end}(V)$.

(iii) $[\,\cdot\,,\,\cdot\,]$ *satisfies the braided derivation property*

$$[\,\cdot\,,\,\cdot\,] \circ (\bullet \otimes \mathrm{id}) = \bullet \circ \Big(\big(\mathrm{id} \otimes [\,\cdot\,,\,\cdot\,]\big) + \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \Phi^{-1} \circ \big(\mathrm{id} \otimes \sigma\big)\Big) \circ \Phi \ , \tag{2.1.41}$$

i.e. $\big[L \circ L', L''\big] = L \circ \big[L', L''\big] + \big[L, L''\big] \circ L'$ *for all* $L, L', L'' \in \mathrm{end}(V)$.

Proof. Item (i) follows from a short calculation

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] &= \bullet - \bullet \circ \sigma = -(\bullet \circ \sigma - \bullet) = -(\bullet - \bullet \circ \sigma^{-1}) \circ \sigma = -(\bullet - \bullet \circ \sigma) \circ \sigma \\ &= -[\,\cdot\,,\,\cdot\,] \circ \sigma \ , \end{aligned} \tag{2.1.42}$$

using the obvious result that $\sigma^{-1} = \sigma$. The proofs of items (ii) and (iii) involve standard manipulations using the associativity of the internal composition. □

Corollary 2.1.14. *Let V be any object in $\mathscr{M}$. Then the object in $\mathscr{M}$ given by the internal endomorphisms* $\mathrm{end}(V)$*, together with the internal commutator* $[\,\cdot\,,\,\cdot\,]$ *given in (2.1.39), is a Lie algebra object in $\mathscr{M}$.*

A commutative algebra in $\mathscr{M}$ is an abelian monoid object (A, μ_A, η_A) in the (symmetric monoidal) category $\mathscr{M}$. In other words

Definition 2.1.15 (Commutative algebra)**.** An algebra (A, μ_A, η_A) in $\mathscr{M}$ is called *commutative* if

$$\begin{tikzcd} A \otimes A \arrow[rr, "\sigma_{A,A}"] \arrow[dr, "\mu_A"'] & & A \otimes A \arrow[dl, "\mu_A"] \\ & A & \end{tikzcd} \tag{2.1.43}$$

in $\mathscr{M}$ commutes. We denote the full subcategory of Alg of commutative algebras in $\mathscr{M}$ by $\mathrm{Alg}^{\mathrm{com}}$.

In the short-hand notation the product in a braided commutative algebra satisfies

$$a\, a' = a'\, a \ , \tag{2.1.44}$$

for all $a, a' \in A$.

2.1.3 Bimodules in $\mathscr{M}$

In what follows we shall simply denote by A the algebra (A, μ_A, η_A) in $\mathscr{M}$. Given an algebra A in $\mathscr{M}$ we can consider objects in $\mathscr{M}$ which are also A-bimodules in a compatible way.

Definition 2.1.16 (Bimodule)**.** Let A be an algebra in $\mathscr{M}$. An A-*bimodule* in $\mathscr{M}$ is an object V in $\mathscr{M}$ together with two $\mathscr{M}$-morphisms $l_V : A \otimes V \to V$ (left A-action) and $r_V : V \otimes A \to V$ (right A-action), such that

$$\begin{tikzcd} (V \otimes A) \otimes A \arrow[r, "r_V \otimes \mathrm{id}_A"] \arrow[d, "\Phi_{V,A,A}"'] & V \otimes A \arrow[dd, "r_V"] \\ V \otimes (A \otimes A) \arrow[d, "\mathrm{id}_V \otimes \mu_A"'] & \\ V \otimes A \arrow[r, "r_V"'] & V \end{tikzcd} \qquad \begin{tikzcd} A \otimes (A \otimes V) \arrow[r, "\mathrm{id}_A \otimes l_V"] \arrow[d, "\Phi^{-1}_{A,A,V}"'] & A \otimes V \arrow[dd, "l_V"] \\ (A \otimes A) \otimes V \arrow[d, "\mu_A \otimes \mathrm{id}_V"'] & \\ A \otimes V \arrow[r, "l_V"'] & V \end{tikzcd} \tag{2.1.45a}$$

$$
\begin{tikzcd}
A\otimes(V\otimes A) \arrow[r, "\mathrm{id}_A\otimes r_V"] \arrow[d, "\Phi^{-1}_{A,V,A}"'] & A\otimes V \arrow[dd, "l_V"] \\
(A\otimes V)\otimes A \arrow[d, "l_V\otimes\mathrm{id}_A"'] & \\
V\otimes A \arrow[r, "r_V"'] & V
\end{tikzcd}
\tag{2.1.45b}
$$

$$
\begin{tikzcd}
I\otimes V \arrow[d, "\eta_A\otimes\mathrm{id}_V"'] \arrow[dr, "\lambda_V"] & \\
A\otimes V \arrow[r, "l_V"'] & V
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & V\otimes I \arrow[d, "\mathrm{id}_V\otimes\eta_A"] \arrow[dl, "\varrho_V"'] \\
V & V\otimes A \arrow[l, "r_V"]
\end{tikzcd}
\tag{2.1.45c}
$$

in $\mathscr{M}$ commute. We shall denote by $\mathrm{Bimod}(A)$ the category with objects all A-bimodules in $\mathscr{M}$ and morphisms given by all structure preserving $\mathscr{M}$-morphisms, i.e. a $\mathrm{Bimod}(A)$-morphism $f : V \to W$ is an $\mathscr{M}$-morphism such that $l_W \circ (\mathrm{id}_A \otimes f) = f \circ l_V$ and $r_W \circ (f \otimes \mathrm{id}_A) = f \circ r_V$.

Given an A-bimodule V in $\mathscr{M}$ it is convenient to denote the left and right A-actions on elements simply by $l_V(a \otimes v) = a\, v$ and $r_V(v \otimes a) = v\, a$, for all $a \in A$ and $v \in V$. In this short-hand notation, the first three diagrams in Definition 2.1.16 imply that

$$
(v\, a)\, a' = v\, (a\, a') \ , \tag{2.1.46a}
$$

$$
a\, (a'\, v) = (a\, a')\, v \ , \tag{2.1.46b}
$$

$$
a\, (v\, a') = (a\, v)\, a' \ , \tag{2.1.46c}
$$

for all $a, a' \in A$ and $v \in V$, and the remaining two diagrams imply that

$$
1_A\, v = v = v\, 1_A \ , \tag{2.1.47}
$$

for all $v \in V$. Then a $\mathrm{Bimod}(A)$-morphism $f : V \to W$ is a k-linear map that satisfies

$$f(a\,v) = a\,f(v) \quad , \quad f(v\,a) = f(v)\,a \quad , \tag{2.1.48}$$

for all $a \in A$ and $v \in V$.

Example 2.1.17. Given any algebra A in $\mathscr{M}$ we can construct the n-dimensional free A-bimodule A^n in $\mathscr{M}$, where $n \in \mathbb{N}$. Elements $\vec{a} \in A^n$ can be written as columns

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad , \qquad a_i \in A \ , \quad i = 1, \dots, n \ . \tag{2.1.49}$$

The left and right A-actions l_{A^n} and r_{A^n} are defined componentwise by

$$a'\,\vec{a} := \begin{pmatrix} a'\,a_1 \\ \vdots \\ a'\,a_n \end{pmatrix} \quad , \quad \vec{a}\,a' := \begin{pmatrix} a_1\,a' \\ \vdots \\ a_n\,a' \end{pmatrix} \quad , \tag{2.1.50}$$

for all $a' \in A$ and $\vec{a} \in A^n$. The A-bimodule properties of A^n follow from the algebra properties of A.

Definition 2.1.18. [Symmetric bimodule] Let A be a braided commutative algebra in $\mathscr{M}$. An A-bimodule V in $\mathscr{M}$ is called *symmetric* if for the left and right A-actions the diagrams

$$\begin{tikzcd} A \otimes V \arrow[rr, "\sigma_{A,V}"] \arrow[dr, "l_V"'] & & V \otimes A \arrow[dl, "r_V"] \\ & V & \end{tikzcd} \qquad \begin{tikzcd} A \otimes V \arrow[dr, "l_V"'] & & V \otimes A \arrow[ll, "\sigma_{V,A}"'] \arrow[dl, "r_V"] \\ & V & \end{tikzcd} \tag{2.1.51}$$

in $\mathscr{M}$ commute. We denote the full subcategory (cf. Definition A.1.3) of $\mathrm{Bimod}(A)$ of symmetric A-bimodules by $\mathrm{Bimod}(A)^{\mathrm{sym}}$.

In the short-hand notation, the left and right A-actions in a symmetric A-

bimodule V in $\mathscr{M}$ satisfy

$$a\,v = v\,a\ , \tag{2.1.52a}$$

$$v\,a = a\,v\ , \tag{2.1.52b}$$

for all $a \in A$ and $v \in V$.

Example 2.1.19. For any braided commutative algebra A in $\mathrm{Alg}^{\mathrm{com}}$ the free A-bimodules of Example 2.1.17 are symmetric A-bimodules.

2.1.4 Monoidal structure on bimodules in $\mathscr{M}$

The monoidal structure on $\mathscr{M}$ induces a monoidal structure $\otimes_A$ (the tensor product over the algebra A) on $\mathrm{Bimod}(A)$ by a construction which we shall now describe. First, by using the forgetful functor $\mathsf{Forget} : \mathrm{Bimod}(A) \to \mathscr{M}$ we can define a functor

$$\otimes \circ (\mathsf{Forget} \times \mathsf{Forget}) : \mathrm{Bimod}(A) \times \mathrm{Bimod}(A) \longrightarrow \mathscr{M}\ . \tag{2.1.53}$$

For any object (V, W) in $\mathrm{Bimod}(A) \times \mathrm{Bimod}(A)$ we can equip the object $V \otimes W$ in $\mathscr{M}$ with the structure of an A-bimodule in $\mathscr{M}$ (here and in the following we suppress the forgetful functors). Let us define the left and right A-action on $V \otimes W$ by the $\mathscr{M}$-morphisms

$$l_{V\otimes W} := \big(l_V \otimes \mathrm{id}_W\big) \circ \Phi^{-1}_{A,V,W} : A \otimes \big(V \otimes W\big) \longrightarrow V \otimes W\ , \tag{2.1.54a}$$

$$r_{V\otimes W} := \big(\mathrm{id}_V \otimes r_W\big) \circ \Phi_{V,W,A} : \big(V \otimes W\big) \otimes A \longrightarrow V \otimes W\ . \tag{2.1.54b}$$

In the short-hand notation, the left and right A-actions on $V \otimes W$ read as

$$l_{V\otimes W}\big(a \otimes (v \otimes w)\big) = (a\,v) \otimes w =: a\,(v \otimes w)\ , \tag{2.1.55a}$$

$$r_{V\otimes W}\big((v \otimes w) \otimes a\big) = v \otimes (w\,a) =: (v \otimes w)\,a\ , \tag{2.1.55b}$$

for all $a \in A$, $v \in V$ and $w \in W$. From these explicit expressions it is immediate that $l_{V\otimes W}$ and $r_{V\otimes W}$ satisfy the properties in Definition 2.1.16 and hence equip

$V \otimes W$ with the structure of an A-bimodule in $\mathscr{M}$. Given a morphism $\big(f : V \to X, g : W \to Y\big)$ in $\mathrm{Bimod}(A) \times \mathrm{Bimod}(A)$, it is also clear from the definitions that the $\mathscr{M}$-morphism $f \otimes g : V \otimes W \to X \otimes Y$ preserves this A-bimodule structure, i.e. it is a morphism in $\mathrm{Bimod}(A)$. As a consequence, the functor in (2.1.53) can be promoted to a functor with values in $\mathrm{Bimod}(A)$, which we shall denote with an abuse of notation by

$$\otimes : \mathrm{Bimod}(A) \times \mathrm{Bimod}(A) \longrightarrow \mathrm{Bimod}(A)~. \tag{2.1.56}$$

We point out some relevant properties which follow directly from the definition of the A-bimodule structure on $\otimes$ and the properties of an A-bimodule in $\mathscr{M}$:

Lemma 2.1.20. *(i) For any three objects V, W, X in* $\mathrm{Bimod}(A)$ *the $\mathscr{M}$-morphism $\Phi_{V,W,X} : (V \otimes W) \otimes X \to V \otimes (W \otimes X)$ is a* $\mathrm{Bimod}(A)$*-morphism with respect to the A-bimodule structure described by the functor (2.1.56).*

(ii) For any object V in $\mathrm{Bimod}(A)$ *the $\mathscr{M}$-morphisms $l_V : A \otimes V \to V$ and $r_V : V \otimes A \to V$ are* $\mathrm{Bimod}(A)$*-morphisms with respect to the A-bimodule structure described by the functor (2.1.56). (In the domain of these morphisms A is regarded as the one-dimensional free A-bimodule, see Example 2.1.17.)*

The functor (2.1.56) is *not yet* the correct monoidal functor on the category $\mathrm{Bimod}(A)$ as it does not take the tensor product over the algebra A. We modify this functor as follows: For any object (V, W) in $\mathrm{Bimod}(A) \times \mathrm{Bimod}(A)$ we have two parallel morphisms in $\mathscr{M}$ given by

$$\begin{tikzcd} (V \otimes A) \otimes W \arrow[r, shift left, "(\mathrm{id}_V \otimes l_W) \circ \Phi_{V,A,W}"] \arrow[r, shift right, "r_V \otimes \mathrm{id}_W"'] & V \otimes W \end{tikzcd}~. \tag{2.1.57}$$

Due to Lemma 2.1.20, the two morphisms in (2.1.57) are $\mathrm{Bimod}(A)$-morphisms. We define the object $V \otimes_A W$ in $\mathrm{Bimod}(A)$ to be the coequalizer of the two parallel $\mathrm{Bimod}(A)$-morphisms (2.1.57), i.e. there is an epimorphism $\pi_{V,W} : V \otimes W \to V \otimes_A W$

in Bimod(A) and the following diagram is a coequaliser

$$(V \otimes A) \otimes W \underset{r_V \otimes \mathrm{id}_W}{\overset{(\mathrm{id}_V \otimes l_W)\circ \Phi_{V,A,W}}{\rightrightarrows}} V \otimes W \xrightarrow{\pi_{V,W}} V \otimes_A W \quad . \tag{2.1.58}$$

We can give an explicit characterization of the coequalizer: Let us denote the image of the difference of the Bimod(A)-morphisms in (2.1.57) by

$$N_{V,W} := \mathrm{Im}\big(r_V \otimes \mathrm{id}_W - (\mathrm{id}_V \otimes l_W) \circ \Phi_{V,A,W}\big) \ , \tag{2.1.59}$$

and notice that $N_{V,W} \subseteq V \otimes W$ is an object in Bimod(A) with respect to the induced A-bimodule structures. Then the object $V \otimes_A W$ in Bimod(A) can be represented explicitly as the quotient

$$V \otimes_A W = \frac{V \otimes W}{N_{V,W}} \ , \tag{2.1.60}$$

and the epimorphism $\pi_{V,W} : V \otimes W \to V \otimes_A W$ is given by the quotient map assigning equivalence classes.

In the spirit of our short-hand notation, we shall denote elements in $V \otimes_A W$ by $v \otimes_A w$, which one should read as the equivalence class in $V \otimes_A W$ defined by the element $v \otimes w \in V \otimes W$, i.e. $v \otimes_A w = \pi_{V,W}(v \otimes w)$. As a consequence of the equivalence relation in $V \otimes_A W$, one has the identity

$$(v\, a) \otimes_A w = v \otimes_A (a\, w) \ , \tag{2.1.61a}$$

for all $a \in A$, $v \in V$ and $w \in W$. The A-bimodule structure on $V \otimes_A W$ in this notation reads as

$$a\,(v \otimes_A w) = (a\, v) \otimes_A w \ , \tag{2.1.61b}$$

$$(v \otimes_A w)\, a = v \otimes_A (w\, a) \ , \tag{2.1.61c}$$

for all $a \in A$, $v \in V$ and $w \in W$.

It can be easily checked that the construction of $V \otimes_A W$ is functorial: Given any morphism $\big(f : V \to X, g : W \to Y\big)$ in $\mathrm{Bimod}(A) \times \mathrm{Bimod}(A)$ we obtain an $\mathrm{Bimod}(A)$-morphism $f \otimes_A g : V \otimes_A W \to X \otimes_A Y$ by setting

$$f \otimes_A g\big(v \otimes_A w\big) := f(v) \otimes_A g(w) \ , \tag{2.1.62}$$

for all $v \in V$ and $w \in W$. We shall denote this functor by

$$\otimes_A : \mathrm{Bimod}(A) \times \mathrm{Bimod}(A) \longrightarrow \mathrm{Bimod}(A) \ . \tag{2.1.63}$$

By Lemma 2.1.20, the (V, W, X)-component $\Phi_{V,W,X}$ of the associator in $\mathscr{M}$ are $\mathrm{Bimod}(A)$-morphisms if V, W, X are in $\mathrm{Bimod}(A)$. With a simple computation using the bimodule properties in Definition 2.1.16 one checks that $\Phi_{V,W,X}$ descends to the quotients for any V, W, X in $\mathrm{Bimod}(A)$, and thereby induce an associator Φ^A for the monoidal functor $\otimes_A$ on $\mathrm{Bimod}(A)$. Explicitly, the (V, W, X)-component of Φ^A reads as

$$\begin{aligned} \Phi^A_{V,W,X} : (V \otimes_A W) \otimes_A X &\longrightarrow V \otimes_A (W \otimes_A X) \ , \\ (v \otimes_A w) \otimes_A x &\longmapsto v \otimes_A (w \otimes_A x) \ . \end{aligned} \tag{2.1.64}$$

Finally, by declaring A (regarded as the one-dimensional free A-bimodule, cf. Example 2.1.17) as the unit object in $\mathrm{Bimod}(A)$, we can define unitors for the monoidal functor $\otimes_A$ on $\mathrm{Bimod}(A)$ by using the fact that $l_V : A \otimes V \to V$ and $r_V : V \otimes A \to V$ are $\mathrm{Bimod}(A)$-morphisms (cf. Lemma 2.1.20) that descend to the quotients (by the bimodule properties in Definition 2.1.16). Explicitly, the V-component of the unitors λ^A and ϱ^A read as

$$\lambda^A_V : A \otimes_A V \longrightarrow V \ , \quad a \otimes_A v \longmapsto a\, v \ , \tag{2.1.65a}$$

$$\varrho^A_V : V \otimes_A A \longrightarrow V \ , \quad v \otimes_A a \longmapsto v\, a \ . \tag{2.1.65b}$$

In summary, this shows

Proposition 2.1.21. *For any algebra A in* Alg*, the category* Bimod(A) *of A-bimodules in $\mathscr{M}$ is a monoidal category with monoidal functor $\otimes_A$ (cf. (2.1.60) and (2.1.62)), associator Φ^A (cf. (2.1.64)), unit object A (regarded as the one-dimensional free A-bimodule, cf. Example (2.1.17), and unitors λ^A and ϱ^A (cf. (2.1.65)).*

Lemma 2.1.22. *Let A be any braided commutative algebra in $\mathscr{M}$. Then the category* Bimod$(A)^{\mathrm{sym}}$ *is a full monoidal subcategory of the monoidal category* Bimod(A)*. Explicitly, the monoidal functor on* Bimod(A) *restricts to the functor (denoted by the same symbol)*

$$\otimes_A : \mathrm{Bimod}(A)^{\mathrm{sym}} \times \mathrm{Bimod}(A)^{\mathrm{sym}} \longrightarrow \mathrm{Bimod}(A)^{\mathrm{sym}} \ . \tag{2.1.66a}$$

Proof. First, notice that the unit object A (regarded as a free A-bimodule) in Bimod(A) is an object in Bimod$(A)^{\mathrm{sym}}$, cf. Example 2.1.19 (i). Next, we shall show that $V \otimes_A W$ is a symmetric A-bimodule for any two objects V, W in Bimod$(A)^{\mathrm{sym}}$. We have

$$\begin{aligned} a\,(v \otimes_A w) &= (a\,v) \otimes_A w \\ &= (v\,a) \otimes_A w \\ &= v \otimes_A (a\,w) \\ &= v \otimes_A (w\,a) \\ &= (v \otimes_A w)\,a \ , \end{aligned} \tag{2.1.67}$$

for all $v \in V$, $w \in W$ and $a \in A$. In the first, third and fifth equalities we used (2.1.61), and in the second and fourth equalities we used (2.1.52). Hence, Bimod$(A)^{\mathrm{sym}}$ is a monoidal subcategory. It is straightforward to see that the restriction of any Bimod(A)-morphism to an object in Bimod$(A)^{\mathrm{sym}}$ is an Bimod$(A)^{\mathrm{sym}}$-morphism and hence Bimod$(A)^{\mathrm{sym}}$ is a full subcategory of Bimod(A). □

2.1.5 Bimodule internal homomorphisms in $\mathscr{M}$

Let A be an algebra in $\mathscr{M}$. Let us consider the monoidal category $\mathrm{Bimod}(A)$ (cf. Proposition 2.1.21) and notice that, by using the forgetful functor Forget : $\mathrm{Bimod}(A) \to \mathscr{M}$, we can define a functor

$$\mathrm{hom} \circ \big(\mathsf{Forget}^{\mathrm{op}} \times \mathsf{Forget}\big) : \big(\mathrm{Bimod}(A)\big)^{\mathrm{op}} \times \mathrm{Bimod}(A) \longrightarrow \mathscr{M} \ . \tag{2.1.68}$$

For any object (V, W) in $\big(\mathrm{Bimod}(A)\big)^{\mathrm{op}} \times \mathrm{Bimod}(A)$ the object $\mathrm{hom}(V, W)$ in $\mathscr{M}$ can be equipped with the structure of an A-bimodule in $\mathscr{M}$ (here and in the following we suppress the forgetful functors). As preparation for this, we require

Lemma 2.1.23. *For any object V in $\mathrm{Bimod}(A)$ the $\mathscr{M}$-morphism*

$$\widehat{l}_V := \zeta_{A,V,V}(l_V) : A \longrightarrow \mathrm{end}(V) \tag{2.1.69}$$

is an Alg-morphism with respect to the algebra structure on $\mathrm{end}(V)$ described in Example 2.1.9.

Proof. Acting with $\widehat{l}_V$ on the unit element $1_A = \eta_A(1) \in A$ and using the expression for the currying map (2.1.14) we obtain

$$\widehat{l}_V(1_A) = l_V(1_A \otimes (\,\cdot\,)) = 1_{\mathrm{end}(V)} \ . \tag{2.1.70}$$

To show that $\widehat{l}_V$ preserves the product, notice that

$$\begin{aligned}
\mu_{\mathrm{end}(V)}\big(\widehat{l}_V(a) \otimes \widehat{l}_V(a'\,)\big) &= \widehat{l}_V(a) \bullet_{V,V,V} \widehat{l}_V(a'\,) \\
&= l_V(a \otimes l_V(a' \otimes (\cdot))) \\
&= (a\,a')\,(\cdot) \\
&= l_V((a\,a') \otimes (\cdot)) \\
&= \widehat{l}_V(a\,a'\,) \ ,
\end{aligned} \tag{2.1.71}$$

for all $a, a' \in A$. Hence $\widehat{l}_V$ is an Alg-morphism. $\square$

Due to Lemma 2.1.23 and the associativity of $\bullet$, the $\mathscr{M}$-morphisms defined by the diagrams

$$\begin{tikzcd} A \otimes \mathrm{hom}(V,W) \arrow[r, "l_{\mathrm{hom}(V,W)}"] \arrow[d, "\widehat{l}_W \otimes \mathrm{id}_{\mathrm{hom}(V,W)}"'] & \mathrm{hom}(V,W) \\ \mathrm{end}(W) \otimes \mathrm{hom}(V,W) \arrow[ur, "\bullet_{V,W,W}"'] & \end{tikzcd} \tag{2.1.72a}$$

$$\begin{tikzcd} \mathrm{hom}(V,W) \otimes A \arrow[r, "r_{\mathrm{hom}(V,W)}"] \arrow[d, "\mathrm{id}_{\mathrm{hom}(V,W)} \otimes \widehat{l}_V"'] & \mathrm{hom}(V,W) \\ \mathrm{hom}(V,W) \otimes \mathrm{end}(V) \arrow[ur, "\bullet_{V,V,W}"'] & \end{tikzcd} \tag{2.1.72b}$$

in $\mathscr{M}$ induce an A-bimodule structure on $\mathrm{hom}(V,W)$. It will be convenient to use the short-hand notation

$$l_{\mathrm{hom}(V,W)}(a \otimes L) = \widehat{l}_W(a) \bullet_{V,W,W} L =: a\, L \ , \tag{2.1.73a}$$

$$r_{\mathrm{hom}(V,W)}(L \otimes a) = L \bullet_{V,V,W} \widehat{l}_V(a) =: L\, a \ , \tag{2.1.73b}$$

for all $a \in A$ and $L \in \mathrm{hom}(V,W)$.

It it useful to note that

Lemma 2.1.24. *The left A-linearity of a* $\mathrm{Bimod}(A)$*-morphism $f : V \to W$, viewed as an internal homomorphism, can be written as*

$$f \bullet \widehat{l}_V(a) = \widehat{l}_W(a) \bullet f \ , \tag{2.1.74}$$

for all $a \in A$.

Given any morphism $\big(f^{\mathrm{op}} : V \to X, g : W \to Y\big)$ in $\big(\mathrm{Bimod}(A)\big)^{\mathrm{op}} \times \mathrm{Bimod}(A)$, the $\mathscr{M}$-morphism $\mathrm{hom}(f^{\mathrm{op}}, g) : \mathrm{hom}(V,W) \to \mathrm{hom}(X,Y)$ preserves the A-bimodule structure on $\mathrm{hom}(V,W)$, hence it is an $\mathrm{Bimod}(A)$-morphism: using equation (2.1.74) and the short-hand notation above, we find that $\mathrm{hom}(f^{\mathrm{op}}, g)$ preserves the left A-

action since

$$\begin{aligned}
\hom(f^{\mathrm{op}}, g)\big(a\, L\big) &= g \circ \big(\widehat{l}_W(a) \bullet_{V,W,W} L\big) \circ f \\
&= (g \bullet_{W,W,Y} \widehat{l}_W(a)) \circ (L \circ f) \\
&= (\widehat{l}_Y(a) \bullet_{W,Y,Y} g) \circ (L \circ f) \\
&= \widehat{l}_Y(a) \bullet_{X,Y,Y} \hom(f^{\mathrm{op}}, g)\big(L\big) \\
&= a \hom(f^{\mathrm{op}}, g)\big(L\big) \ , \qquad (2.1.75)
\end{aligned}$$

for all $a \in A$ and $L \in \hom(V, W)$. By a similar calculation one shows that $\hom(f^{\mathrm{op}}, g)$ preserves the right A-action. As a consequence, the functor in (2.1.68) can be promoted to a functor with values in the category $\mathrm{Bimod}(A)$ which we shall denote with an abuse of notation by

$$\hom : \big(\mathrm{Bimod}(A)\big)^{\mathrm{op}} \times \mathrm{Bimod}(A) \longrightarrow \mathrm{Bimod}(A) \ . \qquad (2.1.76)$$

Intuitively, the internal hom-objects in $\mathrm{Bimod}(A)$ should satisfy conditions which generalise the A-bilinearity of morphisms in $\mathrm{Bimod}(A)$. Now we notice that the condition in (2.1.74) generalises the notion of left A-linearity for internal homomorphisms and that if the monoidal category is symmetric this condition gives automatically also the correct generalisation of right A-linearity. So for the remainder of this section we restrict ourselves to the full subcategory $\mathrm{Bimod}(A)^{\mathrm{sym}}$ of symmetric A-bimodules in $\mathscr{M}$ (recall that in this case A must be an object in $\mathrm{Alg}^{\mathrm{com}}$).

A key observation for this part is that condition (2.1.74) can be translated into a commutator equation.

In the full subcategory $\mathrm{Bimod}(A)^{\mathrm{sym}}$ of symmetric A-bimodules in $\mathscr{M}$ for a braided commutative algebra A in $\mathrm{Alg}^{\mathrm{com}}$, the construction of the internal hom-functor in $\mathrm{Bimod}(A)^{\mathrm{sym}}$ involves a generalisation of the internal commutator $[\,\cdot\,,\,\cdot\,]$ from Definition 2.1.11. For A an object in $\mathrm{Alg}^{\mathrm{com}}$ and V, W any two objects in $\mathrm{Bimod}(A)^{\mathrm{sym}}$ we define an $\mathscr{M}$-morphism (denoted with abuse of notation by the same symbol as the internal commutator) $[\,\cdot\,,\,\cdot\,]_{V,W,A} : \hom(V,W) \otimes A \to \hom(V,W)$

by

$$[\,\cdot\,,\,\cdot\,]_{V,W,A} := \bullet_{V,W} \circ \Big(\big(\mathrm{id}_{\hom(V,W)} \otimes \widehat{l}_V\ \big) - \big(\widehat{l}_W \otimes \mathrm{id}_{\hom(V,W)}\big) \circ \sigma_{\hom(V,W),A} \Big)\ , \tag{2.1.77}$$

where $\widehat{l}$ was defined in (2.1.69). For ease of notation we shall drop indices on the internal commutator in future. Then

$$[L,a] = L \circ \widehat{l}_V(a) - \widehat{l}_W(a) \circ L\ , \tag{2.1.78}$$

for all $L \in \hom(V,W)$ and $a \in A$.

Definition 2.1.25. We define an object $\hom_A(V,W)$ in $\mathscr{M}$ by the equalizer

$$\begin{tikzcd} \hom_A(V,W) \arrow[r] & \hom(V,W) \arrow[r, shift left, "\zeta_{\hom(V,W),A,\hom(V,W)}([\,\cdot\,,\,\cdot\,])"] \arrow[r, shift right, "0"'] & \hom(A,\hom(V,W)) \end{tikzcd} \tag{2.1.79}$$

in $\mathscr{M}$. This equalizer can be realized explicitly in terms of the $\mathscr{M}$-subobject

$$\hom_A(V,W) = \mathrm{Ker}\big(\zeta_{\hom(V,W),A,\hom(V,W)}([\,\cdot\,,\,\cdot\,])\big) \subseteq \hom(V,W) \tag{2.1.80}$$

of the internal hom-object $\hom(V,W)$ in $\mathscr{M}$.

For a simpler characterisation of the object $\hom_A$ in $\mathscr{M}$ we note the following technical

Lemma 2.1.26. *Let V,W,X be any three objects in $\mathscr{M}$. Let $f : V \otimes W \to X$ be any $\mathscr{M}$-morphism. Then $\zeta_{V,W,A}(f) \circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}_W) = 0$, for all $\mathscr{M}$-morphisms $j : U \to V$.*

Proof. Let us first suppose that $\zeta_{V,W,A}(f) \circ j = 0$. Then (dropping indices)

$$0 = \mathrm{ev} \circ \big((\zeta(f) \circ j) \otimes \mathrm{id} \big) = \mathrm{ev} \circ \big(\zeta(f) \otimes \mathrm{id}\big) \circ (j \otimes \mathrm{id}) = f \circ (j \otimes \mathrm{id})\ , \tag{2.1.81}$$

where in the last equality we have used equation (2.1.24a). Let us now assume that

$f \circ (j \otimes \mathrm{id}) = 0$. Then (dropping indices)

$$
\begin{aligned}
0 &= \zeta\big(f \circ (j \otimes \mathrm{id})\big) \\
&= \zeta\big(\mathrm{Hom}_{\mathscr{M}}(j^{\mathrm{op}} \otimes \mathrm{id}^{\mathrm{op}}, \mathrm{id})(f)\big) \\
&= \mathrm{Hom}_{\mathscr{M}}(j^{\mathrm{op}}, \hom(\mathrm{id}^{\mathrm{op}}, \mathrm{id}))\big(\zeta(f)\big) = \zeta(f) \circ j \ ,
\end{aligned} \tag{2.1.82}
$$

where in the third equality we have used naturality of the currying bijection, see equation (2.1.17). □

Lemma 2.1.27. *Let A be any object in $\mathrm{Alg}^{\mathrm{com}}$ and let V, W be any two objects in $\mathrm{Bimod}(A)^{\mathrm{sym}}$. An $\mathscr{M}$-subobject $U \subseteq \hom(V,W)$ is an $\mathscr{M}$-subobject of $\hom_A(V,W)$ if and only if*

$$
[L, a] = 0 \ , \tag{2.1.83}
$$

for all $L \in U$ and $a \in A$.

Proof. We have to show that $\zeta([\,\cdot\,,\,\cdot\,]) \circ j = 0$ if and only if $[\,\cdot\,,\,\cdot\,] \circ (j \otimes \mathrm{id}) = 0$ where $j : U \to \hom(V,W)$ is the inclusion $\mathscr{M}$-morphism. This is a consequence of Lemma 2.1.26 with $f := [\,\cdot\,,\,\cdot\,] : \hom(V,W) \otimes A \to \hom(V,W)$. □

The object $\hom_A(V,W)$ in $\mathscr{M}$ given by (2.1.80) carries a natural left and right A-action given by the $\mathscr{M}$-morphisms (which are the restriction of those in equation (2.1.72) dropping indices)

$$
l := \bullet \circ (\widehat{l} \otimes \mathrm{id}) : A \otimes \hom_A(V,W) \longrightarrow \hom_A(V,W) \ , \tag{2.1.84a}
$$

$$
r := \bullet \circ (\mathrm{id} \otimes \widehat{l}\,) : \hom_A(V,W) \otimes A \longrightarrow \hom_A(V,W) \ . \tag{2.1.84b}
$$

It is moreover an object in $\mathrm{Bimod}(A)^{\mathrm{sym}}$ because the result of Lemma 2.1.27 is precisely the symmetry condition for the left and right A-action given in (2.1.84) (see also (2.1.77)). Moreover, the assignment of the objects $\hom_A(V,W)$ in $\mathrm{Bimod}(A)^{\mathrm{sym}}$

is functorial and we denote the corresponding functor by

$$\mathrm{hom}_A : \big(\mathrm{Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times \mathrm{Bimod}(A)^{\mathrm{sym}} \longrightarrow \mathrm{Bimod}(A)^{\mathrm{sym}} \ . \tag{2.1.85}$$

To any $\big(\mathrm{Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times \mathrm{Bimod}(A)^{\mathrm{sym}}$-morphism $(f^{\mathrm{op}} : V \to V', \ g : W \to W')$ this functor assigns $\mathrm{hom}_A(f^{\mathrm{op}}, g) : \mathrm{hom}_A(V,W) \to \mathrm{hom}_A(V',W')$, $L \mapsto g \circ L \circ f$. Using the biderivation property of the internal commutator (c.f. (2.1.41)) and noticing that an $\mathrm{Bimod}(A)$-morphism f viewed as an internal homomorphism satisfies $[f,a] = 0$ for all $a \in A$, we have that $[g \circ L \circ f, a] = 0$ whenever $[L,a] = 0$, i.e. the image of $\mathrm{hom}_A(f^{\mathrm{op}}, g)(L)$ is contained in $\mathrm{hom}_A(V',W')$ for all $L \in \mathrm{hom}_A(V,W)$. By the same calculation as in (2.1.75), $\mathrm{hom}_A(f^{\mathrm{op}}, g)$ is an $\mathrm{Bimod}(A)$-morphism.

Finally, we show that (2.1.85) is an internal hom-functor in $\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Proposition 2.1.28. *The monoidal category* $\mathrm{Bimod}(A)^{\mathrm{sym}}$ *is closed: There is a natural bijection* $\zeta^A : \mathrm{Hom}_{\mathrm{Bimod}(A)^{\mathrm{sym}}}(- \otimes_A -, -) \Rightarrow \mathrm{Hom}_{\mathrm{Bimod}(A)^{\mathrm{sym}}}(-, \mathrm{hom}_A(-,-))$ *with components given by*

$$\begin{aligned} \zeta^A_{V,W,X}(f) : V &\longrightarrow \mathrm{hom}_A(W,X) \ , \\ v &\longmapsto f\big(v \otimes_A (\,\cdot\,)\big) \ , \end{aligned} \tag{2.1.86}$$

for all $\mathrm{Bimod}(A)^{\mathrm{sym}}$*-morphisms* $f : V \otimes_A W \to X$*. The components of its inverse are*

$$\begin{aligned} (\zeta^A_{V,W,X})^{-1}(g) : V \otimes_A W &\longrightarrow X \ , \\ v \otimes_A w &\longmapsto g(v)(w) \ , \end{aligned} \tag{2.1.87}$$

for all $\mathrm{Bimod}(A)^{\mathrm{sym}}$*-morphisms* $g : V \to \mathrm{hom}_A(W,X)$.

Proof. It is a straightforward calculation to show that the $\mathscr{M}$-morphism $\zeta^A_{V,W,X}(f) : V \to \mathrm{hom}(W,X)$ is a $\mathrm{Bimod}(A)$-morphism, for all $\mathrm{Bimod}(A)$-morphisms $f : V \otimes_A$

$W \to X$. We have

$$\begin{aligned}
\zeta^A_{V,W,X}(f)(a\,v) &= f\big(a\,v \otimes_A (\,\cdot\,)\big) \\
&= f\big(a\,(v \otimes_A (\,\cdot\,))\big) \\
&= a\,f\big(v \otimes_A (\,\cdot\,)\big) \\
&= a\,\zeta^A_{V,W,X}(f)(v)\ ,
\end{aligned} \tag{2.1.88}$$

for all $v \in V$ and $a \in A$. Notice that left A-linearity of $\zeta^A_{V,W,X}(f)$ is a consequence of the left A-linearity of $f : V \otimes_A W \to X$. For right A-linearity we have by the symmetry of the modules $V, \hom_A(W, X)$ that

$$\begin{aligned}
\zeta^A_{V,W,X}(f)(v\,a) &= \zeta^A_{V,W,X}(f)(a\,v) \\
&= a\,\zeta^A_{V,W,X}(f)(v) \\
&= \zeta^A_{V,W,X}(f)(v)\,a\ ,
\end{aligned} \tag{2.1.89}$$

for all $v \in V$ and $a \in A$. Now it must be shown that the image of $\zeta^A(f)$ is contained in $\hom_A(W, X)$ for all $\mathrm{Bimod}(A)^{\mathrm{sym}}$-morphisms $f : V \otimes_A W \to X$. Due to Lemma 2.1.27 this is shown by the calculation

$$\begin{aligned}
\big(\zeta^A_{V,W,X}(f)(v)\big)\,a &= \zeta^A_{V,W,X}(f)(v\,a) \\
&= \zeta^A_{V,W,X}(f)(a\,v) \\
&= a\,\zeta^A_{V,W,X}(f)(v)\ ,
\end{aligned} \tag{2.1.90}$$

for all $a \in A$ and $v \in V$. In the first equality we have used the right A-linearity of $\zeta^A(f)$, in the second equality the symmetry of the A-bimodule V, and in the last equality the left A-linearity of $\zeta^A(f)$.

Next, notice that $(\zeta^A_{V,W,X})^{-1}(g)$ is well-defined as a consequence of the right A-

linearity of g. Indeed

$$
\begin{aligned}
(\zeta^A_{V,W,X})^{-1}(g)(v\,a \otimes_A w) &= g(v\,a)(w) \\
&= (g(v)\,a)(w) \\
&= g(v)(a\,w) \\
&= (\zeta^A_{V,W,X})^{-1}(g)(v \otimes_A a\,w) \ .
\end{aligned} \tag{2.1.91}
$$

It is straightforward to check that left A-linearity of g implies that $(\zeta^A_{V,W,X})^{-1}(g)$ is also left A-linear. Indeed

$$
\begin{aligned}
(\zeta^A_{V,W,X})^{-1}(g)(a\,(v \otimes_A w)) &= g(a\,v)(w) \\
&= (a\,g(v))(w) \\
&= a\,(g(v)(w)) \\
&= a\,(\zeta^A_{V,W,X})^{-1}(g)(v \otimes_A w) \ .
\end{aligned} \tag{2.1.92}
$$

Notice that this calculation also implies that the $\mathscr{M}$-morphism

$$
\mathrm{ev} := \zeta^{-1}(\mathrm{id}_{\hom(V,W)}) \tag{2.1.93}
$$

is left A-linear for $V,W \in \mathrm{Bimod}(A)$. $(\zeta^A)^{-1}(g)$ is also a right A-linear map for all $\mathrm{Bimod}(A)^{\mathrm{sym}}$-morphisms $g : V \to \hom_A(X,Y)$ as is shown by a short calculation

$$
\begin{aligned}
(\zeta^A)^{-1}(g)\big((v \otimes_A w)\,a\big) &= (\zeta^A)^{-1}(g)(v \otimes_A w\,a) \\
&= \mathrm{ev}\big(g(v) \otimes_A (w\,a)\big) \\
&= \mathrm{ev}\big((g(v) \otimes_A w)\,a\big) \\
&= a\,\mathrm{ev}\big(g(v) \otimes_A w\big) \\
&= \big((\zeta^A)^{-1}(g)(v \otimes_A w)\big)\,a \ ,
\end{aligned} \tag{2.1.94}
$$

for all $a \in A$, $v \in V$ and $w \in W$. The second equality holds by direct inspection. In the fourth equality we have used the symmetry of the A-bimodule $\mathrm{hom}_A(V,W)\otimes_A W$ and the fact that the $\mathscr{M}$-morphism ev is left A-linear (c.f. (2.1.92)). The last equality uses the symmetry of the A-bimodule X.

Naturality of ζ^A and the fact that $(\zeta^A_{V,W,X})^{-1}$ is the inverse of $\zeta^A_{V,W,X}$ is easily seen and completely analogous to (2.1.17). □

We conclude this section by showing that $\mathrm{Bimod}(A)^{\mathrm{sym}}$ is a braided closed monoidal category for any braided commutative algebra A in $\mathscr{M}$.

2.1.6 Braiding for bimodules in $\mathscr{M}$

Theorem 2.1.29. *Let A be any braided commutative algebra in $\mathscr{M}$. Then the braiding σ in the closed monoidal category $\mathscr{M}$ descends to a braiding σ^A in the closed monoidal category* $\mathrm{Bimod}(A)^{\mathrm{sym}}$. *Explicitly, the (V,W)-component is*

$$\sigma^A_{V,W} : V\otimes_A W \longrightarrow W\otimes_A V~,\quad v\otimes_A w \longmapsto w\otimes_A v~, \tag{2.1.95}$$

As a consequence, $\mathrm{Bimod}(A)^{\mathrm{sym}}$ *is a braided closed monoidal category.*

Proof. We have to show that (2.1.95) is a well-defined $\mathrm{Bimod}(A)$-morphism, which is equivalent to proving that

$$\pi_{W,V}\circ\sigma_{V,W} : V\otimes W \longrightarrow W\otimes_A V \tag{2.1.96}$$

is a $\mathrm{Bimod}(A)$-morphism that vanishes on $N_{V,W}$ (cf. (2.1.59)). This follows from the symmetry of the left and right A-actions on V,W (c.f. the properties (2.1.52)):

$$\begin{aligned}\pi_{W,V}\circ\sigma_{V,W}(v\otimes a\,w) &= a\,w\otimes_A v\\ &= w\,a\otimes_A v\\ &= w\otimes_A a\,v\\ &= w\otimes_A v\,a\\ &= \pi_{W,V}\circ\sigma_{V,W}(v\,a\otimes w)~,\end{aligned} \tag{2.1.97}$$

for all $a \in A, v \in V$ and $w \in W$. $\square$

2.1.7 Quasi-Hopf algebras

The infinitesimal diffeomorphisms on a classical manifold form a Hopf algebra. The coproduct structure exists because of the Leibniz rule for differentiation, the antipode structure exists because of adjoint action of Lie derivatives and the counit structure exists because of the differentiation of constant functions being trivial (cf. Subsection 4.1).

The nonassociative algebras we consider in this thesis possess a type of nonassociativity structure whose properties are captured exactly by the axioms of a more general type of Hopf algebra-like object called a quasi-Hopf algebra. Quasi-Hopf algebras were first studied by Drinfel'd in [31].

Definition 2.1.30 (Quasi-bialgebra)**.** Let H be an algebra over the ring k with strictly associative product $\mu : H \otimes H \to H$ and unit $\eta : k \to H$. H is a *quasi-bialgebra* if it is further equipped with two algebra homomorphisms $\Delta : H \to H \otimes H$ (coproduct) and $\epsilon : H \to k$ (counit), and an invertible element $\phi \in H \otimes H \otimes H$ (associator), such that

$$(\epsilon \otimes \mathrm{id}_H)\,\Delta(h) = h = (\mathrm{id}_H \otimes \epsilon)\,\Delta(h)\ , \tag{2.1.98a}$$

$$(\mathrm{id}_H \otimes \Delta)\,\Delta(h) \cdot \phi = \phi \cdot (\Delta \otimes \mathrm{id}_H)\,\Delta(h)\ , \tag{2.1.98b}$$

$$(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)(\phi) \cdot (\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi) = (1 \otimes \phi) \cdot (\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_H)(\phi) \cdot (\phi \otimes 1)\ , \tag{2.1.98c}$$

$$(\mathrm{id}_H \otimes \epsilon \otimes \mathrm{id}_H)(\phi) = 1 \otimes 1\ , \tag{2.1.98d}$$

for all $h \in H$.

Remark 2.1.31. In order to simplify the notation, the unit element in H (given by $\eta(1) \in H$) is denoted simply by 1 and the product is written $\mu(h \otimes h') = h \cdot h'$ or simply $h\,h'$. Sweedler notation is used for the coproduct $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$, and the associator is written $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$ and its inverse $\phi^{-1} =$

$\phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$ (with summations understood). If a second copy of the associator is needed its components are decorated with a tilde, e.g. $\phi = \widetilde{\phi}^{(1)} \otimes \widetilde{\phi}^{(2)} \otimes \widetilde{\phi}^{(3)}$. From (2.1.98a), (2.1.98c), (2.1.98d) it follows that

$$(\epsilon \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi) = 1 \otimes 1 = (\mathrm{id}_H \otimes \mathrm{id}_H \otimes \epsilon)(\phi) \ . \tag{2.1.99}$$

Whenever $\phi = 1 \otimes 1 \otimes 1$ the axioms for a quasi-bialgebra reduce to those for a bialgebra.

The antipode structure of a Hopf algebra is modified in a quasi-Hopf algebra to that of a quasi-antipode. The properties of a quasi-antipode enable one to preserve the properties of a quasi-Hopf algebra and its representations under cochain twisting (discussed later in this chapter). The compatibility conditions between the associator and quasi-antipode become important in the definition of the currying bijection for the internal hom-structure in the category of representations of a quasi-Hopf algebra (discussed in Subsection 2.2.5).

Definition 2.1.32 (Quasi-antipode)**.** A *quasi-antipode* for a quasi-bialgebra H is a triple (S, α, β) consisting of an algebra anti-automorphism $S : H \to H$ and two elements $\alpha, \beta \in H$ such that

$$S(h_{(1)})\, \alpha\, h_{(2)} = \epsilon(h)\, \alpha \ , \tag{2.1.100a}$$

$$h_{(1)}\, \beta\, S(h_{(2)}) = \epsilon(h)\, \beta \ , \tag{2.1.100b}$$

$$\phi^{(1)}\, \beta\, S(\phi^{(2)})\, \alpha\, \phi^{(3)} = 1 \ , \tag{2.1.100c}$$

$$S(\phi^{(-1)})\, \alpha\, \phi^{(-2)}\, \beta\, S(\phi^{(-3)}) = 1 \ , \tag{2.1.100d}$$

for all $h \in H$.

Definition 2.1.33 (Quasi-Hopf algebra)**.** A *quasi-Hopf algebra* is a quasi-bialgebra with a quasi-antipode.

Remark 2.1.34. If (S, α, β) is a quasi-antipode for a quasi-bialgebra H and $u \in H$

is any invertible element, then

$$S'(\,\cdot\,) := u\, S(\,\cdot\,)\, u^{-1} \quad , \quad \alpha' := u\, \alpha \quad , \quad \beta' := \beta\, u^{-1} \tag{2.1.101}$$

defines another quasi-antipode (S', α', β') for H. In the case where $\phi = 1 \otimes 1 \otimes 1$ the conditions (2.1.100c,2.1.100d) imply that $\alpha = \beta^{-1}$. Setting $u = \beta$ in (2.1.101) there is an algebra anti-automorphism $S' : H \to H$, which by the conditions (2.1.100a,2.1.100b) satisfies the axioms of an antipode for the bialgebra H. Hence for $\phi = 1 \otimes 1 \otimes 1$ the axioms for a quasi-Hopf algebra reduce to those for a Hopf algebra (up to the transformations (2.1.101) which fix $\alpha = 1 = \beta$).

2.1.8 Quasitriangular structures

The algebras in our motivating examples are not only nonassociative but also noncommutative. They possess a type of noncommutativity structure whose properties are captured exactly by the axioms of a quasitriangular structure on a quasi-Hopf algebra. Quasitriangular structures were first studied by Drinfel'd in [30]. The definitions in the section are taken from [52].

Notation Let H be a quasi-Hopf algebra and $X = X^{(1)} \otimes \cdots \otimes X^{(p)} \in H^{\otimes p}$ (with $p > 1$ and summation understood). For any p-tuple $(i_1, \ldots, i_p)$ of distinct elements of $\{1, \ldots, n\}$ (with $n \geq p$), denote by $X_{i_1,\ldots,i_p}$ the element of $H^{\otimes n}$ given by

$$X_{i_1,\ldots,i_p} = Y^{(1)} \otimes \cdots \otimes Y^{(n)} \qquad \text{(with summation understood)} \quad , \tag{2.1.102}$$

where $Y^{(i_j)} = X^{(j)}$ for $j \in \{1, \cdots, p\}$ and $Y^{(k)} = 1$ otherwise. In other words $X^{(m)}$ is placed in the $i_m^{\rm th}$ position for $m = 1, \cdots, p$ and 1 is placed in all the other positions of the tensor product $X_{i_1,\ldots,i_p} \in H^{\otimes n}$. For example, if $X = X^{(1)} \otimes X^{(2)} \in H^{\otimes 2}$ and $n = 3$, then $X_{12} = X^{(1)} \otimes X^{(2)} \otimes 1 \in H^{\otimes 3}$ and $X_{31} = X^{(2)} \otimes 1 \otimes X^{(1)} \in H^{\otimes 3}$.

Definition 2.1.35 (Quasitriangular quasi-Hopf algebra)**.** A *quasitriangular quasi-Hopf algebra* is a quasi-Hopf algebra H together with an invertible element $R \in$

$H \otimes H$, called the *universal R-matrix*, such that

$$\Delta^{\mathrm{op}}(h) = R\,\Delta(h)\,R^{-1} \ , \tag{2.1.103a}$$

$$(\mathrm{id}_H \otimes \Delta)(R) = \phi_{231}^{-1}\,R_{13}\,\phi_{213}\,R_{12}\,\phi_{123}^{-1} \ , \tag{2.1.103b}$$

$$(\Delta \otimes \mathrm{id}_H)(R) = \phi_{312}\,R_{13}\,\phi_{132}^{-1}\,R_{23}\,\phi_{123} \ , \tag{2.1.103c}$$

for all $h \in H$. Δ^{op} denotes the opposite coproduct, i.e. if $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$, then $\Delta^{\mathrm{op}}(h) = h_{(2)} \otimes h_{(1)}$.

Definition 2.1.36 (Triangular quasi-Hopf algebra)**.** A *triangular quasi-Hopf algebra* is a quasitriangular quasi-Hopf algebra such that

$$R_{21} = R^{-1} \ . \tag{2.1.104}$$

For brevity in this thesis the adjective 'universal' is dropped and R is simply referred to as an R-matrix. It will be convenient to denote the R-matrix $R \in H \otimes H$ of a quasitriangular quasi-Hopf algebra by $R = R^{(1)} \otimes R^{(2)}$ and its inverse by $R^{-1} = R^{(-1)} \otimes R^{(-2)}$ (with summations understood).

Remark 2.1.37. Whenever H is a quasitriangular quasi-Hopf algebra with R-matrix $R \in H \otimes H$, then $R' := R_{21}^{-1} \in H \otimes H$ is also an R-matrix, i.e. it satisfies the conditions in (2.1.103) using that $\Delta(h)_{21} = \Delta^{\mathrm{op}}(h)$, $(\mathrm{id}_H \otimes \Delta)(R_{21}^{-1}) = [(\Delta \otimes \mathrm{id}_H)(R)]_{312}^{-1}$ and $(\Delta \otimes \mathrm{id}_H)(R_{21}^{-1}) = [(\mathrm{id}_H \otimes \Delta)(R)]_{312}^{-1}$. If H is a triangular quasi-Hopf algebra then the two R-matrices R and R' coincide, cf. (2.1.104).

Lemma 2.1.38. *If H is a quasitriangular quasi-Hopf algebra with R-matrix $R \in H \otimes H$, then*

$$(\epsilon \otimes \mathrm{id})(R) = 1 = (\mathrm{id} \otimes \epsilon)(R) \ . \tag{2.1.105}$$

Proof. We have $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = R_{23}$ by (2.1.98a), and also that $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13})\,R_{23}$ by (2.1.103c) and (2.1.99). Since R_{23} is invertible, it follows that $(\epsilon \otimes \mathrm{id})(R) = 1$. By a similar calculation with

$(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\mathrm{id} \otimes \Delta)(R)$ we obtain $(\mathrm{id} \otimes \epsilon)(R) = 1$. □

2.1.9 Cochain twisting of quasi-Hopf algebras

It is possible to modify the structures of a quasi-Hopf algebra (in particular its coproduct and antipode structures) in such a way that the deformed object continues to satisfy the axioms of a quasi-Hopf algebra. This is done via the method of cochain twisting which we shall describe in this section. Cochain twisting of quasi-Hopf algebras is defined in such a way that the representation categories of cochain twist related quasi-Hopf algebras are equivalent. This will be shown in the next section, but here we collect definitions for future reference.

The theory of cochain twisting of quasi-Hopf algebras is explained in terms of a cohomology structure in [52], but the concept was first described by Drinfel'd in [31].

Definition 2.1.39 (Cochain twist)**.** A *cochain twist* based on a quasi-Hopf algebra H is an invertible element $F \in H \otimes H$ satisfying

$$\big(\epsilon \otimes \mathrm{id}_H\big)(F) = 1 = \big(\mathrm{id}_H \otimes \epsilon\big)(F)\ . \tag{2.1.106}$$

It is convenient to introduce the following notation: a cochain twist shall be denoted by $F = F^{(1)} \otimes F^{(2)} \in H \otimes H$ and its inverse by $F^{-1} = F^{(-1)} \otimes F^{(-2)} \in H \otimes H$ (with summations understood). We note that $F^{(1)}$, $F^{(2)}$, $F^{(-1)}$ and $F^{(-2)}$ are elements in H. Then the counital condition (2.1.106) reads as

$$\epsilon(F^{(1)})\, F^{(2)} = 1 = \epsilon(F^{(2)})\, F^{(1)} \tag{2.1.107a}$$

and its inverse reads as

$$\epsilon(F^{(-1)})\, F^{(-2)} = 1 = \epsilon(F^{(-2)})\, F^{(-1)}\ . \tag{2.1.107b}$$

The following result is due to Drinfel'd in [31].

Theorem 2.1.40 (Twisting of Hopf algebras)**.** *Given any cochain twist $F \in H \otimes H$ based on a quasi-Hopf algebra H there is a new quasi-Hopf algebra H_F. As an algebra, H_F equals H, and they also have the same counit $\epsilon_F := \epsilon$. The coproduct in H_F is given by*

$$\Delta_F(\,\cdot\,) := F\,\Delta(\,\cdot\,)\,F^{-1} \tag{2.1.108}$$

and the associator in H_F reads as

$$\phi_F := (1 \otimes F) \cdot (\mathrm{id}_H \otimes \Delta)(F) \cdot \phi \cdot (\Delta \otimes \mathrm{id}_H)(F^{-1}) \cdot (F^{-1} \otimes 1)\ . \tag{2.1.109}$$

The quasi-antipode (S_F, α_F, β_F) in H_F is given by $S_F := S$ and

$$\alpha_F := S(F^{(-1)})\,\alpha\,F^{(-2)} \quad , \quad \beta_F := F^{(1)}\,\beta\,S(F^{(2)})\ . \tag{2.1.110}$$

Proof. This result can be seen with a direct check of the relations (2.1.98) and (2.1.100) for the quasi-Hopf algebra H_F (i.e. in (2.1.98) and (2.1.100) one has to replace Δ by Δ_F, ϕ by ϕ_F, α by α_F and β by β_F). The proof involves elementary calculations making use of the corresponding conditions for the untwisted quasi-Hopf algebra and properties of the cochain twist, see e.g. [43, Proposition XV.3.2]. □

Remark 2.1.41. If F is any cochain twist based on H, then its inverse F^{-1} is a cochain twist based on the quasi-Hopf algebra H_F. By twisting H_F with the cochain twist F^{-1} one obtains the original quasi-Hopf algebra H, i.e. $(H_F)_{F^{-1}} = H$. More generally, if F is any cochain twist based on H and G is any cochain twist based on H_F, then the product $G\,F$ is a cochain twist based on H and $H_{G\,F} = (H_F)_G$. (See B.1 for details of a proof.)

This is a very important result for this thesis; the fact that cochain twisting defines an equivalence of categories (discussed in the next section) is a consequence of this result.

Remark 2.1.42. If H is a Hopf algebra, i.e. $\phi = 1 \otimes 1 \otimes 1$, and F is a cochain twist based on H, then in general H_F is a quasi-Hopf algebra since ϕ_F need not be trivial. The condition that H_F is again a Hopf algebra, i.e. that also $\phi_F = 1 \otimes 1 \otimes 1$, is equivalent to the 2-cocycle condition on F

$$(1 \otimes F) \cdot (\mathrm{id}_H \otimes \Delta)(F) \cdot (\Delta \otimes \mathrm{id}_H)(F^{-1}) \cdot (F^{-1} \otimes 1) = 1 \ , \tag{2.1.111}$$

and in this case F is called a *cocycle twist* based on H. The noncommutative but strictly associative spaces discussed in the motivation of this thesis are representations of a quasitriangular Hopf algebra. The twists which perform the deformation quantisation in these examples satisfy the cocycle condition above.

When a (quasi-)Hopf algebra possesses the additional structure of a quasitriangular structure, then the R-matrix is also modified under the cochain twist. The following result is due to Drinfel'd [30], [31].

Theorem 2.1.43 (Twisting of quasitriangular (quasi-)Hopf Algebras)**.** *If $F \in H \otimes H$ is any cochain twist based on a quasitriangular quasi-Hopf algebra H with R-matrix $R \in H \otimes H$, then the quasi-Hopf algebra H_F of Theorem 2.1.40 is quasitriangular with R-matrix*

$$R_F := F_{21}\, R\, F^{-1} \ . \tag{2.1.112}$$

Moreover, H_F is triangular if and only if H is triangular.

Proof. Similarly to the proof of Theorem 2.1.40, the first part of the proof can be seen with a direct check of the relations (2.1.103) for R_F in the quasi-Hopf algebra H_F (i.e. in (2.1.103) replacing Δ by Δ_F, ϕ by ϕ_F and R by R_F). For the second part, notice that $(R_F)_{21} = F\, R_{21}\, F_{21}^{-1}$ and $R_F^{-1} = F\, R^{-1}\, F_{21}^{-1}$, hence $(R_F)_{21} = R_F^{-1}$ if and only if $R_{21} = R^{-1}$ since F is invertible (and hence can be cancelled from the equation). □

2.2 A quasi-Hopf representation category

In this section we study the representation category $[H, \mathscr{M}]$ of a quasitriangular quasi-Hopf algebra H in the category $\mathscr{M}$ of k-modules. The properties of a quasitriangular quasi-Hopf algebra are such that the structures on $\mathscr{M}$ discussed in Section 2.1 descend to structures on the category $[H,\mathscr{M}]$. That is $[H,\mathscr{M}]$ is also a closed braided monoidal category: it admits a monoidal structure as well as an internal hom-functor and can be equipped with a braiding. We show the important result for physics that the morphisms in $[H,\mathscr{M}]$ are contained in the internal homomorphisms in a way that preserves the map-like structures of composition and tensor product. We also consider algebras ρ_A in the monoidal category $[H,\mathscr{M}]$. Furthermore given any cochain twist based on H, we deform the quasitriangular quasi-Hopf algebra H into a new quasitriangular quasi-Hopf algebra H_F, and show that $[H,\mathscr{M}]$ and $[H_F,\mathscr{M}]$ are equivalent as closed braided monoidal categories. It is important to note that although these categories are equivalent, the physical models built out of the structures in the categories are not equivalent; the selection criteria for physically realisable data from the models based on different categories would be different. We also show that the assignment of twisted algebras ρ_{A_F} in $[H_F,\mathscr{M}]$ to algebras ρ_A in $[H,\mathscr{M}]$ is functorial.

2.2.1 The quasi-Hopf representation category

Definition 2.2.1 (Representation)**.** Let H be a quasi-Hopf algebra over k and let V be an object in the category $\mathscr{M}$. A *representation* of H on V is an Alg-morphism

$$\rho_V : H \longrightarrow \mathrm{end}(V) \ . \tag{2.2.1}$$

In particular

$$\rho_V(h\,h') = \rho_V(h)\,\rho_V(h') \ , \rho_V(1) = \mathrm{id}_{\mathrm{end}(V)} \ , \tag{2.2.2}$$

for all $h,h' \in H$.

Using the currying bijection in $\mathscr{M}$

$$\zeta : \mathrm{Hom}_{\mathscr{M}}(H\otimes V,V) \longrightarrow \mathrm{Hom}_{\mathscr{M}}(H,\mathrm{end}(V)) \ , \tag{2.2.3}$$

we can define the $\mathscr{M}$-morphism

$$\triangleright_V := \zeta^{-1}(\rho_V) : H \otimes V \longrightarrow V \ . \tag{2.2.4}$$

An equivalent but different perspective on the notion of representation of H, and which is more conventional in the physics literature, is the notion of H-action.

Definition 2.2.2. [H-action] Let H be a quasi-Hopf algebra in $\mathscr{M}$ and let V be an object in $\mathscr{M}$. An *action of H or H-action* on V is an $\mathscr{M}$-morphism (conventionally written with infix notation)

$$\triangleright_V : H \otimes V \to V \ , \ \ h \otimes v \mapsto h \triangleright_V v \ , \tag{2.2.5}$$

such that for all $h, h' \in H$ and $v \in V$

$$h \triangleright_V \big(h' \triangleright_V v\big) = \big(h\, h'\,\big) \triangleright_V v \quad , \qquad 1 \triangleright_V v = v \ . \tag{2.2.6}$$

Remark 2.2.3. Although it is less cumbersome to use the infix action notation, it is more convenient for computational purposes to use representation notation because it is less amenable to inserting elements of objects than the action notation is. The representation notation also makes it more evident that proofs only entail the manipulation of properties of a quasitriangular quasi-Hopf algebra and its representations. This is an advantage in that one is able to see the abstract structures involved more clearly. In Chapter 4, which is aimed at a physics audience, we use mainly the conventional action notation.

Definition 2.2.4 (Representation category of H)**.** Viewing a quasi-Hopf algebra in $\mathscr{M}$ as a one-object category (with morphisms given by elements of H and composition of morphisms given by the product in H) the *representation category of the quasi-Hopf algebra H* is the functor category (cf. Definition A.2.10)

$$[H, \mathscr{M}] \ . \tag{2.2.7}$$

In the language of actions, $[H, \mathscr{M}]$ is the category of *left H-modules.*

In the rest of this section we show how the structures together with their properties on the closed braided monoidal category $\mathscr{M}$ discussed in Section 2.1 descend to the category $[H,\mathscr{M}]$.

Objects in $[H,\mathscr{M}]$ are functors: For an object V in $\mathscr{M}$ we denote, with an abuse of notation, the corresponding functor in $[H,\mathscr{M}]$ by ρ_V. Denoting by $*$ the single object of the category H, the functor ρ_V is defined by

$$\rho_V(*) := V \ , \quad \rho_V(h) := \triangleright_V(h,-) \ , \tag{2.2.8}$$

for any $h \in H$ where $\triangleright_V$ is an H-action on V (cf. Definition 2.2.2). Functoriality of $\triangleright_V$ corresponds to the representation properties (2.2.2). Morphisms in $[H,\mathscr{M}]$ are natural transformations

$$f : \rho_V \Longrightarrow \rho_W \ , \tag{2.2.9}$$

with single component given by the $\mathscr{M}$-morphism (denoted with abuse of notation by the same symbol)

$$f : V \longrightarrow W \ , \tag{2.2.10}$$

satisfying the naturality condition

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(h)"'] & W \arrow[d, "\rho_W(h)"] \\ V \arrow[r, "f"'] & W \end{tikzcd} \tag{2.2.11}$$

for any $h \in H$. This property is called H-*equivariance* of the k-linear map f and on elements $v \in V$ it reads as

$$f\big(\rho_V(h)\, v\big) = \rho_W(h)\, f(v) \ , \tag{2.2.12}$$

for all $h \in H$.

2.2.2 Cochain twisting the category

We consider a cochain twist F based on H and twist H to the quasi-Hopf algebra H_F in the manner described in theorem 2.1.40 and note that any left H-module is also a left H_F-module as the H_F-action on an H_F-module is only sensitive to the algebra structure underlying H_F, which agrees with that of H. So there is a functor

$$\mathcal{F} : [H, \mathscr{M}] \longrightarrow [H_F, \mathscr{M}] \ , \tag{2.2.13}$$

between the representation categories of H and H_F, defined by

$$\mathcal{F}(\rho_V) = \rho_V \ , \quad \mathcal{F}(f) = f \ , \tag{2.2.14}$$

for any object $V \in \mathscr{M}$ and any natural transformation f in $[H, \mathscr{M}]$. We shall denote by $\mathcal{F}(\rho_V)(*) = \mathcal{F}(V)$ or $\mathcal{F}(\rho_V)(*) = V_F$ the corresponding object in $\mathscr{M}$.

Theorem 2.2.5. *If H is a quasi-Hopf algebra and $F \in H \otimes H$ is any cochain twist based on H, then $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$ are equivalent as categories.*

Proof. From (2.1.107b) and the definition of a cochain twist it is clear that $F^{-1} \in H_F \otimes H_F$ is a cochain twist based on H_F. So we can define a functor $\mathcal{F}^{-1} : [H_F, \mathscr{M}] \to [H, \mathscr{M}]$ in the same way as above. Using that $(H_F)_{F^{-1}} = H$ and $(H_{F\,F^{-1}})_F = H_F$, cf. Remark 2.1.41, we have that $\mathcal{F} \circ \mathcal{F}^{-1} \cong \mathrm{id}_{[H_F, \mathscr{M}]}$ and $\mathcal{F}^{-1} \circ \mathcal{F} \cong \mathrm{id}_{[H, \mathscr{M}]}$. □

2.2.3 The monoidal structure

The monoidal structure in $\mathscr{M}$ is modified in the following way in $[H, \mathscr{M}]$: We define a functor $\otimes : [H, \mathscr{M}] \times [H, \mathscr{M}] \to [H, \mathscr{M}]$ (denoted with abuse of notation by the same symbol as the monoidal functor on the category $\mathscr{M}$) as follows: For any object (ρ_V, ρ_W) in $[H, \mathscr{M}] \times [H, \mathscr{M}]$

$$\begin{aligned} &\rho_V \otimes \rho_W(*) := V \otimes W \ , \\ &\rho_V \otimes \rho_W(h) := (\rho_V \otimes \rho_W)(\Delta(h)) = \rho_V(h_{(1)}) \otimes \rho_V(h_{(2)}) \ , \end{aligned} \tag{2.2.15}$$

for any $h \in H$ where $V \otimes W$ is the tensor product of the underlying k-modules. This is a representation of H because Δ is an algebra map (c.f. Definition 2.1.30). Indeed

$$\begin{aligned}
\rho_V \otimes \rho_W\,(h\,k) &= (\rho_V \otimes \rho_W)(\Delta(h\,k)) \\
&= \rho_V(h_{(1)}\,k_{(1)}) \otimes \rho_W(h_{(2)}\,k_{(2)}) \\
&= \big(\rho_V(h_{(1)}) \otimes \rho_W(h_{(2)})\big)\,\big(\rho_V(k_{(1)}) \otimes \rho_W(k_{(2)})\big) \\
&= \big(\rho_V \otimes \rho_W\,(h)\big)\,\big(\rho_V \otimes \rho_W\,(k)\big)\ ,
\end{aligned}$$

for any $h, k \in H$, and

$$\begin{aligned}
\rho_V \otimes \rho_W\,(1_H) &= \rho_V(1_H) \otimes \rho_W(1_H) \\
&= \mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V\otimes W}\ . \qquad (2.2.16)
\end{aligned}$$

Having shown that $\rho_V \otimes \rho_W$ is an object in $[H, \mathscr{M}]$ corresponding to the tensor product object $V \otimes W$ in $\mathscr{M}$ we may write

$$\rho_{V\otimes W} := \rho_V \otimes \rho_W\ . \qquad (2.2.17)$$

For a morphism $\big(f : \rho_V \Rightarrow \rho_X, g : \rho_W \Rightarrow \rho_Y\big)$ in $[H, \mathscr{M}] \times [H, \mathscr{M}]$ we set

$$f \otimes g : \rho_V \otimes \rho_W \Longrightarrow \rho_X \otimes \rho_Y\ , \qquad (2.2.18)$$

with single component the tensor product k-linear map $f \otimes g : V \otimes W \to X \otimes Y$ given by the monoidal structure on $\mathscr{M}$ (c.f. (2.1.1)). The H-equivariance of f and g ensures that the k-linear map $f \otimes g$ is H-equivariant and hence a morphism in

$[H,\mathscr{M}]$. Indeed, for any $v \in V$, $w \in W$ and $h \in H$

$$\begin{aligned} f \otimes g\,(\rho_{V\otimes W}(h)\,(v \otimes w)) &= f(\rho_V(h_{(1)})\,v) \otimes g(\rho_W(h_{(2)})\,w) \\ &= \rho_X(h_{(1)})\,f(v) \otimes \rho_Y(h_{(2)})\,g(w) \\ &= (\rho_{X\otimes Y}(h)\,(f \otimes g\,(v \otimes w))\ . \end{aligned} \tag{2.2.19}$$

The unit object in $[H,\mathscr{M}]$ is the functor ρ_I defined by

$$\rho_I(*) := k\ , \quad \rho_I(h) := \epsilon(h)\ , \tag{2.2.20}$$

for any $h \in H$. This is a representation of H since ϵ is an algebra map (c.f. Definition 2.1.30). Indeed $\rho_I(h\,k) = \epsilon(h\,k) = \epsilon(h)\,\epsilon(k) = \rho_I(h)\,\rho_I(k)$ for any $h,k \in H$, and $\rho_I(1_H) = \epsilon(1_H) = 1_k = \mathrm{id}_k$.

The associator $\Phi : \otimes \circ (\otimes \times \mathrm{id}_{[H,\mathscr{M}]}) \Rightarrow \otimes \circ (\mathrm{id}_{[H,\mathscr{M}]} \times \otimes)$ in $[H,\mathscr{M}]$ is given in terms of the associator ϕ in the quasi-Hopf algebra H by the natural transformation with (ρ_V,ρ_W,ρ_X)-component

$$\Phi_{\rho_V,\rho_W,\rho_X} : (\rho_V \otimes \rho_W) \otimes \rho_X \Longrightarrow \rho_V \otimes (\rho_W \otimes \rho_X)\ , \tag{2.2.21}$$

whose single component is the k-linear map

$$\Phi_{V,W,X} := \big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)(\phi)\ . \tag{2.2.22}$$

The naturality condition follows by the coassociativity condition (2.1.98b) and the functoriality of representations. The pentagon relations for Φ follow from the 3-cocycle condition (2.1.98c). (See B.2 for proofs.)

The unitors λ and ϱ in the monoidal category $\mathscr{M}$ canonically induce unitors $\lambda : \rho_I \otimes - \Rightarrow \mathrm{id}_{[H,\mathscr{M}]}$ and $\varrho : - \otimes \rho_I \Rightarrow \mathrm{id}_{[H,\mathscr{M}]}$ in $[H,\mathscr{M}]$ with ρ_V-components the natural transformations

$$\lambda_{\rho_V} : \rho_I \otimes \rho_V \Longrightarrow \rho_I \ , \quad \varrho_{\rho_V} : \rho_V \otimes \rho_I \Longrightarrow \rho_I \ , \tag{2.2.23}$$

whose single components are given by the k-linear maps $\lambda_V = \lambda_V$ and $\varrho_V = \varrho_V$ in $\mathscr{M}$. The H-equivariance for (2.2.23) follow from the condition (2.1.98a) and the k-linearity of representations. The triangle relations for λ and ϱ follow from the counital condition (2.1.98d). (See B.2 for proofs.)

In summary,

Proposition 2.2.6. *For any quasi-Hopf algebra H the category $[H, \mathscr{M}]$ of left H-modules is a monoidal category.*

Remark 2.2.7. If H is a Hopf algebra, i.e. $\phi = 1 \otimes 1 \otimes 1$, then the components of Φ are identity maps and $[H, \mathscr{M}]$ is a *strict* monoidal category.

2.2.4 Cochain twisting the monoidal structure

The functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ between the representation categories of H and H_F is a monoidal functor. To keep track of which quasi-Hopf algebra is acting, we denote the monoidal functor on $[H_F, \mathscr{M}]$ by $\otimes_F$.

The coherence maps for the monoidal functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ are given by the natural isomorphism $\varphi : \otimes_F \circ (\mathcal{F} \otimes \mathcal{F}) \Rightarrow \mathcal{F} \circ \otimes$ of functors from $[H, \mathscr{M}] \times [H, \mathscr{M}]$ to $[H_F, \mathscr{M}]$, with (ρ_V, ρ_W)-component

$$\varphi_{\rho_V, \rho_W} : \mathcal{F}(\rho_V) \otimes_F \mathcal{F}(\rho_W) \Longrightarrow \mathcal{F}(\rho_V \otimes \rho_W) \ , \tag{2.2.24a}$$

and the natural isomorphism

$$\psi : \rho_{I_F} \Longrightarrow \mathcal{F}(\rho_I) \ . \tag{2.2.24b}$$

The single component of φ_{ρ_V, ρ_W} is the k-linear map

$$\varphi_{V,W} := (\rho_V \otimes \rho_W)(F^{-1}) \ , \tag{2.2.25}$$

with inverse given by replacing F^{-1} with F. Naturality (H-equivariance) holds by the calculation

$$\begin{aligned}(\rho_V \otimes \rho_W)(\Delta(h)) \circ \varphi_{V,W} &= (\rho_V \otimes \rho_W)(\Delta(h))(\rho_V \otimes \rho_W)(F^{-1}) \\ &= (\rho_V \otimes \rho_W)(F^{-1})(\rho_V \otimes \rho_W)(\Delta_F(h)) \\ &= \varphi_{V,W} \circ (\rho_V \otimes \rho_W)(\Delta_F(h)) \ ,\end{aligned} \tag{2.2.26}$$

using (2.1.108). Since the inverse of a cochain twist is a cochain twist it is evident by a similar calculation to that in (2.2.26) that the inverse of φ_{ρ_V,ρ_W} is also an $[H_F, \mathscr{M}]$-morphism. The single component of ψ is the identity map on k (since the counit in H_F is equal to that in H).

It is a straightforward check using the counital condition (2.1.107b) that the coherence diagrams (denoting the unitors in $[H_F, \mathscr{M}]$ by λ^F and ϱ^F)

$$\begin{tikzcd} \mathcal{F}(\rho_V) \otimes_F \rho_{I_F} \arrow[r, Rightarrow, "\mathrm{id}_{\mathcal{F}(\rho_V)} \otimes_F \psi"] \arrow[d, Rightarrow, "\varrho^F_{\mathcal{F}(\rho_V)}"'] & \mathcal{F}(\rho_V) \otimes_F \mathcal{F}(\rho_I) \arrow[d, Rightarrow, "\varphi_{\rho_V,\rho_I}"] \\ \mathcal{F}(\rho_V) & \mathcal{F}(\rho_V \otimes \rho_I) \arrow[l, Rightarrow, "\mathcal{F}(\varrho_{\rho_V})"] \end{tikzcd} \tag{2.2.27a}$$

$$\begin{tikzcd} \rho_{I_F} \otimes_F \mathcal{F}(\rho_V) \arrow[r, Rightarrow, "\psi \otimes_F \mathrm{id}_{\mathcal{F}(\rho_V)}"] \arrow[d, Rightarrow, "\lambda^F_{\mathcal{F}(\rho_V)}"'] & \mathcal{F}(\rho_I) \otimes_F \mathcal{F}(\rho_V) \arrow[d, Rightarrow, "\varphi_{\rho_I,\rho_V}"] \\ \mathcal{F}(\rho_V) & \mathcal{F}(\rho_I \otimes \rho_V) \arrow[l, Rightarrow, "\mathcal{F}(\lambda_{\rho_V})"] \end{tikzcd} \tag{2.2.27b}$$

in $H, \mathscr{M}$ commute for any object ρ_V in $[H, \mathscr{M}]$. Indeed by the inverse of (2.1.106)

$$\begin{aligned}\mathcal{F}(\lambda_{\rho_V}) \circ \varphi_{I,V} \circ (\psi \otimes_F \mathrm{id}_{\mathcal{F}(\rho_V)}) &= \mathcal{F}(\lambda_{\rho_V}) \circ (\rho_I \otimes \rho_V)(F^{-1}) \circ (\mathrm{id}_{\rho_{I_F}} \otimes_F \mathrm{id}_{\mathcal{F}(\rho_V)}) \\ &= \lambda_V \circ (\mathrm{id}_I \otimes \rho_V)(\epsilon \otimes \mathrm{id}_H)(F^{-1}) \\ &= \lambda_V \circ (\mathrm{id}_{\rho_I} \otimes \rho_V(1_H)) \\ &= \lambda_V = \lambda^F_{\mathcal{F}(V)} \ .\end{aligned} \tag{2.2.27c}$$

And similarly for the first diagram. Furthermore, by the definition of the associator in H_F in terms of the associator in H (2.1.109) the coherence diagram (denoting the associator in $[H_F, \mathscr{M}]$ by Φ^F)

$$\begin{tikzcd}
\big(\mathcal{F}(\rho_V)\otimes_F \mathcal{F}(\rho_W)\big)\otimes_F \mathcal{F}(\rho_X) \arrow[r, Rightarrow, "\Phi^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W),\mathcal{F}(\rho_X)}"] \arrow[d, Rightarrow, "\varphi_{\rho_V,\rho_W}\otimes_F \mathrm{id}_{\mathcal{F}(\rho_X)}"'] & \mathcal{F}(\rho_V)\otimes_F \big(\mathcal{F}(\rho_W)\otimes_F \mathcal{F}(\rho_X)\big) \arrow[d, Rightarrow, "\mathrm{id}_{\mathcal{F}(\rho_V)}\otimes_F \varphi_{\rho_W,\rho_X}"] \\
\mathcal{F}(\rho_V\otimes\rho_W)\otimes_F \mathcal{F}(\rho_X) \arrow[d, Rightarrow, "\varphi_{\rho_V\otimes\rho_W,\rho_X}"'] & \mathcal{F}(\rho_V)\otimes_F \mathcal{F}(\rho_W\otimes\rho_X) \arrow[d, Rightarrow, "\varphi_{\rho_V,\rho_W\otimes\rho_X}"] \\
\mathcal{F}\big((\rho_V\otimes\rho_W)\otimes\rho_X\big) \arrow[r, Rightarrow, "\mathcal{F}(\Phi_{\rho_V,\rho_W,\rho_X})"'] & \mathcal{F}\big(\rho_V\otimes(\rho_W\otimes\rho_X)\big)
\end{tikzcd} \tag{2.2.27d}$$

commutes for any three objects ρ_V, ρ_W, ρ_X in $[H, \mathscr{M}]$. Indeed, we have by (2.2.25), (2.2.22) and (2.1.109) that

$$\begin{aligned}
&\varphi_{V,W\otimes X}\circ\big(\mathrm{id}_{\mathcal{F}(V)}\otimes_F \varphi_{W,X}\big)\circ\Phi^F_{\mathcal{F}(V),\mathcal{F}(W),\mathcal{F}(X)}\\
&=(\rho_V\otimes(\rho_W\otimes\rho_X))((\mathrm{id}_H\otimes\Delta)(F^{-1})\cdot(1\otimes F^{-1})\cdot\phi_F)\\
&=((\rho_V\otimes\rho_W)\otimes\rho_X)(\phi\cdot(\Delta\otimes\mathrm{id}_H)(F^{-1})\cdot(F^{-1}\otimes 1))\\
&=\mathcal{F}(\Phi_{V,W,X})\circ\varphi_{V\otimes W,X}\circ\big(\varphi_{V,W}\otimes_F \mathrm{id}_{\mathcal{F}(X)}\big)\ .
\end{aligned} \tag{2.2.28}$$

Note that the above diagrams commute with arrows reversed too because of the property that the inverse of a cochain twist is a cochain twist. This proves

Theorem 2.2.8. *If H is a quasi-Hopf algebra and $F \in H\otimes H$ is any cochain twist, then $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$ are equivalent as monoidal categories.*

2.2.5 The internal hom-structure

The internal hom-structure in $\mathscr{M}$ is modified in the following way in $[H, \mathscr{M}]$: For any object (ρ_V, ρ_W) in $[H, \mathscr{M}]^{\mathrm{op}}\times[H, \mathscr{M}]$ we set

$$\mathrm{hom}(\rho_V,\rho_W)(*) := \mathrm{Hom}_{\mathscr{M}}(V,W)\ , \tag{2.2.29a}$$

$$\begin{aligned}
\mathrm{hom}(\rho_V,\rho_W)(h) &:= \rho_W(h_{(1)})\circ\mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)}(-)\circ\rho_V(S(h_{(2)}))\\
&= \circ^3\big((\rho_W\otimes\mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)}\otimes\rho_V)\big([(1\otimes S)\cdot\Delta(h)]_{13}\big)\big)\ ,
\end{aligned} \tag{2.2.29b}$$

for any $h \in H$ where $\mathrm{Hom}_{\mathscr{M}}(V, W)$ is the k-module of k-linear maps between the k-modules V and W and we have denoted by $\circ^3$ the (associative) composition of three k-linear maps.

We note that H is represented via the adjoint representation on internal hom-objects in $[H, \mathscr{M}]$ and the adjoint representation makes use of the quasi-antipode in H.

Lemma 2.2.9. *If (ρ_V, ρ_W) is an object in $[H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}]$ then* $\mathrm{hom}(\rho_V, \rho_W)$ *is an object in $[H, \mathscr{M}]$.*

Proof. It must be shown that (2.2.29b) defines a representation of H, i.e. that the conditions in (2.2.2) are satisfied. This is a consequence of the fact that $\rho_V \otimes \rho_W$ is a representation, that the coproduct Δ is an algebra map and that the antipode S is an anti-algebra map: From $\Delta(1) = 1 \otimes 1$ and $S(1) = 1$ and the representation property of $\rho_V \otimes \rho_W$ we obtain the second equality and from

$$\begin{aligned}(1 \otimes S)\big(\Delta(h\,h')\big) &= (1 \otimes S) = (h\,h'\,)_{(1)} \otimes S((h\,h'\,)_{(2)}) \\ &= h_{(1)}\,\big(h'_{(1)} \otimes S(h'_{(2)})\big) S(h_{(2)}) \\ &= (1 \otimes S)\big(\Delta(h)\big) \cdot (1 \otimes S)\big(\Delta(h')\big)\;, \end{aligned} \tag{2.2.30}$$

for all $h, h' \in H$ and the representation property of $\rho_V \otimes \rho_W$ together with the result that $[XY]_{13} = X_{13} Y_{13}$ for any $X, Y \in H^{\otimes 3}$, we obtain the first equality. $\square$

Having now shown that $\mathrm{hom}(\rho_V, \rho_W)$ is an object in $[H, \mathscr{M}]$ corresponding to the object $\mathrm{hom}(V, W)$ in $\mathscr{M}$ we may write

$$\rho_{\mathrm{hom}(V,W)} := \mathrm{hom}(\rho_V, \rho_W)\;. \tag{2.2.31}$$

Given now any morphism $\big(f^{\mathrm{op}} : \rho_V \Rightarrow \rho_X, g : \rho_W \Rightarrow \rho_Y\big)$ in $[H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}]$, the map of functors

$$\mathrm{hom}(f^{\mathrm{op}}, g) : \mathrm{hom}(\rho_V, \rho_W) \Longrightarrow \mathrm{hom}(\rho_X, \rho_Y)\;, \tag{2.2.32}$$

with single component the k-linear map

$$\mathrm{hom}(f^{\mathrm{op}}, g) : \mathrm{Hom}_{\mathscr{M}}(V, W) \longrightarrow \mathrm{Hom}_{\mathscr{M}}(X, Y) \quad , \quad L \longmapsto g \circ L \circ f \ . \tag{2.2.33}$$

is an $[H, \mathscr{M}]$-morphism. Naturality is a consequence of the naturality of f and g:

$$\begin{aligned} &(\rho_W \otimes \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)} \otimes \rho_V)\big([(1 \otimes S) \cdot \Delta(h)]_{13}\big) \circ \big(g \otimes - \otimes f\big) \\ &= (g \otimes - \otimes f) \circ (\rho_W \otimes - \otimes \rho_V)\big([(1 \otimes S) \cdot \Delta(h)]_{13}\big) \ . \end{aligned} \tag{2.2.34}$$

Lemma 2.2.10. *There is a functor*

$$\mathrm{hom} : [H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}] \longrightarrow [H, \mathscr{M}] \ , \tag{2.2.35}$$

defined on objects by (2.2.29) *and on morphisms by* (2.2.33)*.*

Proof. We have shown in Lemma 2.2.9 and (2.2.34) that hom assigns to objects (resp. morphisms) in $[H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}]$ objects (resp. morphisms) in $[H, \mathscr{M}]$. Functoriality of hom follows from that of $\mathrm{Hom}_{\mathscr{M}}$ (cf. (2.1.12)). $\square$

With these preparations one can now show that $[H, \mathscr{M}]$ is a closed monoidal category.

Theorem 2.2.11. *For any quasi-Hopf algebra H the representation category $[H, \mathscr{M}]$ is a closed monoidal category with internal hom-functor* $\mathrm{hom} : [H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}] \to [H, \mathscr{M}]$ *described above.*

Proof. The currying map in $\mathscr{M}$ is modified in the following way in $[H, \mathscr{M}]$: For any three objects ρ_V, ρ_W, ρ_X in $[H, \mathscr{M}]$ we define the map of functors

$$\zeta_{\rho_V, \rho_W, \rho_X} : \mathrm{Hom}_{\mathscr{M}}\big(\rho_V \otimes \rho_W, \rho_X\big) \Longrightarrow \mathrm{Hom}_{\mathscr{M}}\big(\rho_V, \mathrm{hom}(\rho_W, \rho_X)\big) \tag{2.2.36}$$

on any $[H, \mathscr{M}]$-morphism $f : \rho_V \otimes \rho_W \Rightarrow \rho_X$ by

$$\zeta_{\rho_V, \rho_W, \rho_X}(f) : \rho_V \Longrightarrow \mathrm{hom}(\rho_W, \rho_X) \ , \tag{2.2.37}$$

with single component $\zeta_{V,W,X}(f) : V \to \hom(W,X)$ defined by

$$\begin{aligned}\zeta_{V,W,X}(f) &:= f \circ \big(\rho_V(\phi^{(-1)}) \otimes \rho_W(\phi^{(-2)}\,\beta\, S(\phi^{(-3)}))\big)\\ &= f \circ (\rho_V \otimes \rho_W) \circ (1 \otimes \mu_H^3)\big([(1 \otimes 1 \otimes S)(\phi^{-1})]_{124}\beta_3\big)\ ,\end{aligned} \tag{2.2.38}$$

where we have denoted by μ_H^3 the (associative) multiplication of three elements in H. The map $\zeta_{V,W,X}(f)$ is clearly k-linear. Naturality holds by the following calculation (dropping for computational purposes the triple compositions $\circ^3$ and multiplications μ_H^3 and adding a $\hat{}$ on indices in their stead)

$$\begin{aligned}\hom(\rho_W,\rho_X)(h) \circ \zeta_{V,W,X}(f) &= (\rho_X \otimes \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(W,X)} \otimes \rho_W)\big([(1 \otimes S)\cdot \Delta(h)]_{1\hat{3}}\big)\circ\\ &\quad \circ f \circ (\rho_V \otimes \rho_W)\big([(1 \otimes 1 \otimes S)(\phi^{-1})]_{1\hat{2}4}\,\beta_{\hat{3}}\big)\\ &= f \circ (\rho_V \otimes \rho_W)\big[(1 \otimes 1 \otimes S)(\Delta \otimes \mathrm{id}) \circ \Delta(h) \cdot \phi^{-1}\big]_{1\hat{2}4}\,\beta_{\hat{3}}\\ &= f \circ (\rho_V \otimes \rho_W)\big([(1 \otimes 1 \otimes S)(\phi^{-1})]_{1\hat{2}4}\,\beta_{\hat{3}}\big)\ \circ \rho_V(h)\\ &= \zeta_{V,W,X}(f) \circ \rho_V(h)\ ,\end{aligned} \tag{2.2.39}$$

where the second equality follows from the naturality of f and the third from the calculation $[(1 \otimes 1 \otimes S) \cdot (\Delta \otimes \mathrm{id}_H) \circ \Delta(h) \cdot \phi^{-1}]_{1\hat{2}4}\beta_{\hat{3}} = [(1 \otimes 1 \otimes S) \cdot \phi^{-1} \cdot (\mathrm{id}_H \otimes \Delta) \circ \Delta(h))]_{1\hat{2}4}\beta_{\hat{3}} = (\phi^{(-1)}\, h_{(1)} \otimes \phi^{(-2)}\, h_{(2)_{(1)}}\, \beta\, S(h_{(2)_{(2)}})\, S(\phi^{(-3)})) = (\phi^{(-1)}\, h \otimes \phi^{(-2)}\, \beta\, S(\phi^{(-3)})) = [(1 \otimes 1 \otimes S)(\phi^{-1})]_{1\hat{2}4}\beta_{\hat{3}}\,(h \otimes 1_H)$ using the coassociativity of the coproduct (2.1.98b), the property that S is an algebra anti-automorphism and the property (2.1.100b) of the quasi-antipode together with the property (2.1.98d) of the associator. Hence (2.2.36) is an $[H,\mathscr{M}]$-morphism.

The inverse $\zeta^{-1}_{\rho_V,\rho_W,\rho_X} : \mathrm{Hom}_{\mathscr{M}}(\rho_V, \hom(\rho_W,\rho_X)) \to \mathrm{Hom}_{\mathscr{M}}(\rho_V \otimes \rho_W, \rho_X)$ is given on any $[H,\mathscr{M}]$-morphism $g : \rho_V \Rightarrow \hom(\rho_W,\rho_X)$ by the natural transformation

$$\zeta^{-1}_{\rho_V,\rho_W,\rho_X}(g) : \rho_V \otimes \rho_W \Longrightarrow \rho_X\ , \tag{2.2.40}$$

with single component $\zeta^{-1}_{V,W,X}(g) : V \otimes W \to X$ defined by

$$\begin{aligned}\zeta^{-1}_{V,W,X}(g) &:= \rho_X(\phi^{(1)}) \circ g(-) \circ \rho_W(S(\phi^{(2)})\alpha\phi^{(3)}) \\ &= \circ^3\Big((\rho_X \otimes g(-) \otimes \rho_W)\big(\big[(1 \otimes \mu_H^3)\big([(1 \otimes S \otimes 1)(\phi)]_{124}\,\alpha_3\big)\big]_{13}\big)\Big)\ .\end{aligned} \tag{2.2.41}$$

A calculation similar to that in (2.2.39) shows that $\zeta^{-1}_{\rho_V,\rho_W,\rho_X}(g)$ is an $[H,\mathscr{M}]$-morphism. That $\zeta^{-1}_{\rho_V,\rho_W,\rho_X}$ is the inverse of $\zeta_{\rho_V,\rho_W,\rho_X}$ follows from

$$\begin{aligned}\zeta(\zeta^{-1}(g)) &= \zeta^{-1}(g) \circ \big(\rho_V(\phi^{(-1)}) \otimes \rho_W(\phi^{(-2)}\,\beta\,S(\phi^{(-3)}))\big) \\ &= \rho_X(\widetilde{\phi}^{(1)}) \circ g\big(\rho_V(\phi^{(-1)})\big) \circ \rho_W\big(S(\widetilde{\phi}^{(2)})\alpha\widetilde{\phi}^{(3)}\,\phi^{(-2)}\,\beta\,S(\phi^{(-3)})\big) \\ &= \rho_X(\widetilde{\phi}^{(1)}\,\phi^{(1)}_{(1)}) \circ g(-) \circ \rho_W(S(\widetilde{\phi}^{(2)}\,\phi^{(1)}_{(2)})\,\alpha\,\widetilde{\phi}^{(3)}\phi^{(-2)}\,\beta\,S(\phi^{(-3)})) \\ &= g\ ,\end{aligned} \tag{2.2.42}$$

where in the third equality we have used the H-equivariance of g and in the final equality we have used the properties (2.1.98c), (2.1.100a), (2.1.100b) and (2.1.100d) and the property that S is an anti-algebra morphism, and

$$\begin{aligned}\zeta^{-1}(\zeta(f)) &= \zeta^{-1}\big(f \circ \big(\rho_V(\phi^{(-1)}) \otimes \rho_W(\phi^{(-2)}\,\beta\,S(\phi^{(-3)}))\big)\big) \\ &= \rho_X(\widetilde{\phi}^{(1)}) \circ f \circ \big(\rho_V(\phi^{(-1)}) \otimes \rho_W(\phi^{(-2)}\,\beta\,S(\phi^{(-3)}))\big) \circ \rho_W(S(\widetilde{\phi}^{(2)})\,\alpha\,\widetilde{\phi}^{(3)}) \\ &= f \circ \big(\rho_V(\widetilde{\phi}^{(1)}_{(1)}\,\phi^{(-1)}) \otimes \rho_W(\widetilde{\phi}^{(1)}_{(2)}\,\phi^{(-2)}\,\beta\,S(\widetilde{\phi}^{(2)}\,\phi^{(-3)})\,\alpha\,\widetilde{\phi}^{(3)})\big) \\ &= f\ ,\end{aligned} \tag{2.2.43}$$

where in the third equality we have used the H-equivariance of f and in the final equality we have used the properties (2.1.98c), (2.1.100a),(2.1.100b) and (2.1.100c) and the property that S is an anti-algebra morphism.

It remains to prove naturality, which means that $\zeta_{\rho_V,\rho_W,\rho_X}$ is the (ρ_V,ρ_W,ρ_X)-component of a natural isomorphism ζ between the two functors $\mathrm{Hom}_{\mathscr{M}}(-\otimes -,-)$ and $\mathrm{Hom}_{\mathscr{M}}(-,\mathrm{hom}(-,-))$ from $[H,\mathscr{M}]^{\mathrm{op}} \times [H,\mathscr{M}]^{\mathrm{op}} \times [H,\mathscr{M}]$ to the category of sets.

Explicitly, given any morphism $\big(f_V^{\mathrm{op}} : \rho_V \Rightarrow \rho'_V, f_W^{\mathrm{op}} : \rho_W \Rightarrow \rho'_W, f_X : \rho_X \Rightarrow \rho'_X\big)$ in $[H,\mathscr{M}]^{\mathrm{op}} \times [H,\mathscr{M}]^{\mathrm{op}} \times [H,\mathscr{M}]$ one has to show that the diagram (in the category Sets)

$$\begin{tikzcd}
\mathrm{Hom}\big(\rho_V \otimes \rho_W, \rho_X\big) \arrow[r, Rightarrow, "\zeta_{\rho_V,\rho_W,\rho_X}"] \arrow[d, Rightarrow, "\mathrm{Hom}(f_V^{\mathrm{op}}\otimes f_W^{\mathrm{op}}, f_X)"'] & \mathrm{Hom}\big(\rho_V, \mathrm{hom}(\rho_W,\rho_X)\big) \arrow[d, Rightarrow, "\mathrm{Hom}(f_V^{\mathrm{op}}, \mathrm{hom}(f_W^{\mathrm{op}}, f_X))"] \\
\mathrm{Hom}\big(\rho'_V \otimes \rho'_W, \rho'_X\big) \arrow[r, Rightarrow, "\zeta_{\rho'_V,\rho'_W,\rho'_X}"'] & \mathrm{Hom}\big(\rho'_V, \mathrm{hom}(\rho'_W,\rho'_X)\big)
\end{tikzcd} \tag{2.2.44}$$

commutes. For any $[H,\mathscr{M}]$-morphism $f : \rho_V \otimes \rho_W \Rightarrow \rho_X$ one obtains

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{M}}(f_V^{\mathrm{op}}, \mathrm{hom}(f_W^{\mathrm{op}}, f_X))\big(\zeta_{V,W,X}(f)\big) \\
&\qquad = f_X \circ \Big(f \circ (\rho_V \otimes \rho_W)\big([(1\otimes 1\otimes S)(\phi^{-1})]_{1\hat{2}4}\beta_{\hat{3}}\big) \circ f_V\Big) \circ f_W \\
&\qquad = f_X \circ f \circ (f_V \otimes f_W) \circ (\rho_V \otimes \rho_W)\big([(1\otimes 1\otimes S)(\phi^{-1})]_{1\hat{2}4}\beta_{\hat{3}}\big) \\
&\qquad = \zeta_{V',W',X'}\big(\mathrm{Hom}(f_V^{\mathrm{op}} \otimes f_W^{\mathrm{op}}, f_X)(f)\big) \ ,
\end{aligned} \tag{2.2.45}$$

where the second equality follows from H-equivariance of both f_V and f_W. □

The definition of the currying bijection for the internal hom-structure in $[H,\mathscr{M}]$ can be found in [52]. However it is useful to understand where this definition comes from. We give an explanation at the end of the following section on the cochain twisting of the internal hom-structure.

2.2.6 Cochain twisting the internal hom-structure

Given any cochain twist $F \in H \otimes H$ based on H, we denote the internal hom-functor on $[H_F,\mathscr{M}]$ by hom_F. One can define for any object (ρ_V,ρ_W) in $[H,\mathscr{M}]^{\mathrm{op}} \times [H,\mathscr{M}]$ a map of functors

$$\gamma_{\rho_V,\rho_W} : \mathrm{hom}_F\big(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)\big) \Longrightarrow \mathcal{F}\big(\mathrm{hom}(\rho_V,\rho_W)\big) \ , \tag{2.2.46}$$

with single component $\gamma_{V,W} : \mathrm{Hom}_{\mathscr{M}}\big(V,W\big) \to \mathrm{Hom}_{\mathscr{M}}(V,W)$ given by

$$\gamma_{V,W} := \circ^3\big((\rho_W \otimes \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)} \otimes \rho_V)\big([(1 \otimes S) \cdot F^{-1}]_{13}\big)\big) \ . \tag{2.2.47}$$

γ_{ρ_V,ρ_W} is an $[H_F, \mathscr{M}]$-isomorphism: The inverse of (2.2.47) is given by replacing F^{-1} with F and for any $h \in H$

$$\begin{aligned}\gamma_{V,W} \circ \mathrm{hom}_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))(h) &= (\rho_W \otimes \mathrm{id} \otimes \rho_V)\big([(1 \otimes S) \cdot F^{-1}\Delta_F(h)]_{\hat{13}}\big)\\ &= (\rho_W \otimes \mathrm{id} \otimes \rho_V)\big([(1 \otimes S) \cdot \Delta(h)F^{-1}]_{\hat{13}}\big)\\ &= \mathrm{hom}(\rho_V, \rho_W)(h) \circ \gamma_{V,W} \ ,\end{aligned} \tag{2.2.48}$$

using that $S_F = S$ and $\Delta_F(h) = F\,\Delta(h)\,F^{-1}$ for all $h \in H$. Since the inverse of a cochain twist is a cochain twist it is evident that the inverse of γ_{ρ_V,ρ_W} is also an $[H_F, \mathscr{M}]$-morphism.

A straightforward calculation shows that γ_{ρ_V,ρ_W} is the (ρ_V, ρ_W)-component of a natural isomorphism $\gamma : \mathrm{hom}_F \circ (\mathcal{F}^{\mathrm{op}} \times \mathcal{F}) \Rightarrow \mathcal{F} \circ \mathrm{hom}$ of functors from $[H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}]$ to $[H_F, \mathscr{M}]$. The coherence diagram

$$\begin{tikzcd}
\mathrm{hom}_F\big(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)\big) \arrow[r, Rightarrow, "\gamma_{\rho_V,\rho_W}"] \arrow[d, Rightarrow, "\mathrm{hom}_F(\mathcal{F}^{\mathrm{op}}(f^{\mathrm{op}}),\mathcal{F}(g))"'] & \mathcal{F}\big(\mathrm{hom}(\rho_V, \rho_W)\big) \arrow[d, Rightarrow, "\mathcal{F}(\mathrm{hom}(f^{\mathrm{op}},g))"] \\
\mathrm{hom}_F\big(\mathcal{F}(\rho_X), \mathcal{F}(\rho_Y)\big) \arrow[r, Rightarrow, "\gamma_{\rho_X,\rho_Y}"'] & \mathcal{F}\big(\mathrm{hom}(\rho_X, \rho_Y)\big)
\end{tikzcd} \tag{2.2.49}$$

commutes for all morphisms $\big(f^{\mathrm{op}} : \rho_V \Rightarrow \rho_X, g : \rho_W \Rightarrow \rho_Y\big)$ in $[H, \mathscr{M}]^{\mathrm{op}} \times [H, \mathscr{M}]$. Indeed by the H-equivariance of f and g

$$\begin{aligned}&\mathcal{F}(\mathrm{hom}(f^{\mathrm{op}}, g)) \circ \gamma_{X,Y}\\ &= \circ^3\big((g \otimes - \otimes f) \circ (\rho_W \otimes \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)} \otimes \rho_V)\big([(1 \otimes S) \cdot F^{-1}]_{13}\big)\big)\\ &= \circ^3\big((\rho_W \otimes \mathrm{id}_{\mathrm{Hom}_{\mathscr{M}}(V,W)} \otimes \rho_V)\big([(1 \otimes S) \cdot F^{-1}]_{13}\big) \circ (g \otimes - \otimes f)\big)\\ &= \gamma_{V,W} \circ \mathrm{hom}_F(\mathcal{F}^{\mathrm{op}}(f^{\mathrm{op}}), \mathcal{F}(g)) \ .\end{aligned} \tag{2.2.50}$$

Note that the diagram above also commutes with arrows reversed. This shows

Theorem 2.2.12. *If H is a quasi-Hopf algebra and $F \in H \otimes H$ is any cochain twist based on H, then $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$ are equivalent as closed monoidal categories.*

We can gain insight into the way in which the currying map for the internal hom-functor in $[H, \mathscr{M}]$ is constructed by considering how the currying map for the internal hom-functor in $[H_F, \mathscr{M}]$ arises as described below, where F is a cochain twist based on H. We consider the sequence of morphisms necessary to rebracket the following tensor product in $[H_F, \mathscr{M}]$ by using the coherence maps for the monoidal and internal hom-structures between the representation categories of H and H_F and using the associator in H:

$$\begin{array}{c}
\mathcal{F}(V) \otimes_F \big(\mathrm{hom}_F(\mathcal{F}(W), \mathcal{F}(X)) \otimes_F \mathcal{F}(W)\big) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes_F(\gamma_{W,X}\otimes_F \mathrm{id})} \\
\mathcal{F}(V) \otimes_F \big(\mathcal{F}(\mathrm{hom}(W,X)) \otimes_F \mathcal{F}(W)\big) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes_F \varphi_{\mathrm{hom}(W,X),W}} \\
\mathcal{F}(V) \otimes_F \mathcal{F}\big(\mathrm{hom}(W,X) \otimes W\big) \\
\Big\downarrow{\scriptstyle \varphi_{V,\mathrm{hom}(W,X)\otimes W}} \\
\mathcal{F}\big(V \otimes (\mathrm{hom}(W,X) \otimes W)\big) \\
\Big\downarrow{\scriptstyle \Phi_{-1}} \\
\mathcal{F}\big((V \otimes \mathrm{hom}(W,X)) \otimes W\big) \\
\Big\downarrow{\scriptstyle \varphi^{-1}_{V\otimes\mathrm{hom}(W,X),W}} \\
\mathcal{F}\big(V \otimes \mathrm{hom}(W,X)\big) \otimes_F \mathcal{F}(W) \\
\Big\downarrow{\scriptstyle \varphi^{-1}_{V,\mathrm{hom}(W,X)}\otimes_F \mathrm{id}} \\
\big(\mathcal{F}(V) \otimes \mathcal{F}(\mathrm{hom}(W,X))\big) \otimes_F \mathcal{F}(W)
\end{array} \tag{2.2.51}$$

Recalling that representations of H on internal hom-objects contain the antipode, the above sequence of morphisms corresponds to the following element of H

$$(\mathrm{id} \otimes \mu_H)\big[(1 \otimes 1 \otimes S)(F \otimes 1) \cdot (\Delta \otimes 1)(F) \cdot \phi^{-1} \cdot (1 \otimes \Delta)(F^{-1}) \cdot (1 \otimes F^{-1})\big]_{124}$$
$$\cdot\, (1 \otimes S)(F)_3 = \phi_F^{(-1)} \otimes_F \phi_F^{(-2)}\, \beta_F\, S(\phi_F^{(-3)}) \;,$$

where ϕ_F^{-1} and $\beta_F = F^{(1)}\, \beta\, S(F^{(2)})$ (with $\beta = 1$ here) were defined in (2.1.109) and

(2.1.110). This gives the core content of the currying map defined in (2.2.38). For the inverse we note that by the properties of the quasi-antipode and associator in H_F we have

$$\big(\widetilde{\phi}_F^{(1)}\big)\,\phi_F^{(-1)}\big(S(\widetilde{\phi}_F^{(2)}\,\alpha_F\,\widetilde{\phi}_F^{(3)})\big)\phi_F^{(-2)}\,\beta_F\,S(\phi_F^{(-3)}) = 1\ , \tag{2.2.52}$$

where $\alpha_F = S(F^{(-1)})\,\alpha\,F^{(-2)}$ with $\alpha = 1$ here (cf. the calculation (2.2.42)) and we have that $\widetilde{\phi}_F^{(1)} \otimes_F S(\widetilde{\phi}_F^{(2)}\,\alpha_F\,\widetilde{\phi}_F^{(3)})$ is the core content of the inverse currying map defined in (2.2.41).

2.2.7 Evaluation and composition

For any two objects ρ_V, ρ_W in the monoidal category $[H, \mathscr{M}]$, we calculate from Proposition 2.1.5 and (2.2.41) the internal evaluation $\mathrm{ev}_{V,W} : \mathrm{hom}(\rho_V, \rho_W) \otimes \rho_V \Rightarrow \rho_W$ to be

$$\mathrm{ev}_{V,W} = \zeta^{-1}_{\mathrm{hom}(V,W),V,W}\big(\mathrm{id}_{\mathrm{hom}(V,W)}\big) = \rho_W(\phi^{(1)}) \circ (-) \circ \rho_V\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\big)\ . \tag{2.2.53}$$

We recall from Proposition 2.1.5 that for any three objects ρ_V, ρ_W, ρ_X in $[H, \mathscr{M}]$ the internal composition $\bullet_{V,W,X} : \mathrm{hom}(\rho_W, \rho_X) \otimes \mathrm{hom}(\rho_V, \rho_W) \Rightarrow \mathrm{hom}(\rho_V, \rho_X)$ is defined in terms of the internal evaluations by

$$\begin{aligned}&\bullet_{V,W,X} := \\ &\quad \zeta_{\mathrm{hom}(W,X)\otimes\mathrm{hom}(V,W),V,X}\Big(\mathrm{ev}_{W,X} \circ \big(\mathrm{id}_{\mathrm{hom}(W,X)} \otimes \mathrm{ev}_{V,W}\big) \circ \Phi_{\mathrm{hom}(W,X),\mathrm{hom}(V,W),V}\Big)\ .\end{aligned} \tag{2.2.54}$$

The properties of the internal evaluation and composition morphisms given in (2.1.24) are modified as follows in $[H, \mathscr{M}]$: (here and in the following subscripts are occasionally dropped for ease of notation)

Proposition 2.2.13. *Let H be a quasi-Hopf algebra.*

(i) For any three objects ρ_V, ρ_W, ρ_X in $[H, \mathscr{M}]$ and any $[H, \mathscr{M}]$-morphism g :

$\rho_V \Rightarrow \hom(\rho_W, \rho_X)$ *the diagram*

$$\begin{tikzcd}
\rho_V \otimes \rho_W \arrow[r, Rightarrow, "g\otimes\mathrm{id}"] \arrow[dr, Rightarrow, "\zeta^{-1}(g)"'] & \rho_{\hom(W,X)} \otimes \rho_W \arrow[d, Rightarrow, "\mathrm{ev}"] \\
& \rho_X
\end{tikzcd} \tag{2.2.55}$$

commutes. That is

$$\mathrm{ev} \circ (g \otimes \mathrm{id}) = \zeta^{-1}(g) \ . \tag{2.2.56}$$

(ii) For any three objects ρ_V, ρ_W, ρ_X in $[H, \mathscr{M}]$ the diagram

$$\begin{tikzcd}
\big(\rho_{\hom(W,X)} \otimes \rho_{\hom(V,W)}\big) \otimes \rho_V \arrow[rr, Rightarrow, "\bullet\otimes\mathrm{id}"] \arrow[d, Rightarrow, "\Phi"'] & & \rho_{\hom(V,X)} \otimes \rho_V \arrow[dd, Rightarrow, "\mathrm{ev}"] \\
\rho_{\hom(W,X)} \otimes \big(\rho_{\hom(V,W)} \otimes \rho_V\big) \arrow[d, Rightarrow, "\mathrm{id}\otimes\mathrm{ev}"'] & & \\
\rho_{\hom(W,X)} \otimes \rho_W \arrow[rr, Rightarrow, "\mathrm{ev}"'] & & \rho_X
\end{tikzcd} \tag{2.2.57}$$

commutes. That is

$$\mathrm{ev} \circ (\bullet \otimes \mathrm{id}) = \mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \Phi \ , \tag{2.2.58}$$

(iii) The composition morphisms are weakly associative, i.e. for any four objects $\rho_V, \rho_W, \rho_X, \rho_Y$ in $[H, \mathscr{M}]$ the diagram

$$\begin{tikzcd}
\big(\rho_{\hom(X,Y)} \otimes \rho_{\hom(W,X)}\big) \otimes \rho_{\hom(V,W)} \arrow[rr, Rightarrow, "\bullet\otimes\mathrm{id}"] \arrow[d, Rightarrow, "\Phi"'] & & \rho_{\hom(W,Y)} \otimes \rho_{\hom(V,W)} \arrow[dd, Rightarrow, "\bullet"] \\
\rho_{\hom(X,Y)} \otimes \big(\rho_{\hom(W,X)} \otimes \rho_{\hom(V,W)}\big) \arrow[d, Rightarrow, "\mathrm{id}\otimes\bullet"'] & & \\
\rho_{\hom(X,Y)} \otimes \rho_{\hom(V,X)} \arrow[rr, Rightarrow, "\bullet"'] & & \rho_{\hom(V,Y)}
\end{tikzcd} \tag{2.2.59}$$

commutes. That is

$$\bullet \circ (\bullet \otimes \mathrm{id}) = \bullet \circ (\mathrm{id} \otimes \bullet) \circ \Phi \ . \tag{2.2.60}$$

Proof. The commutative diagram in item (i) follows directly from the definitions (2.2.53) and (2.2.41):

$$\begin{aligned}
\mathrm{ev}_{W,X} \circ (g \otimes \mathrm{id}_W) &= \rho_X(\phi^{(1)}) \circ (-) \circ \rho_W\big(S(\phi^{(2)})\, \alpha\, \phi^{(3)}\big) \circ (g \otimes \mathrm{id}_W) \\
&= \rho_X(\phi^{(1)}) \circ g(-) \circ \rho_W\big(S(\phi^{(2)})\, \alpha\, \phi^{(3)}\big) \\
&= \zeta^{-1}_{V,W,X}(g) \ .
\end{aligned} \tag{2.2.61}$$

Item (ii) follows from item (i) and the definition of the internal composition (2.2.54):

$$\begin{aligned}
\mathrm{ev}_{V,X} \circ (\bullet_{V,W,X} \otimes \mathrm{id}_V) &= \zeta^{-1}_{\mathrm{hom}(W,X)\otimes \mathrm{hom}(V,W),V,X}(\bullet_{V,W,X}) \\
&= \mathrm{ev}_{W,X} \circ (\mathrm{id}_{\mathrm{hom}(W,X)} \otimes \mathrm{ev}_{V,W}) \circ \Phi_{\mathrm{hom}(W,X),\mathrm{hom}(V,W),V} \ .
\end{aligned} \tag{2.2.62}$$

In order to prove item (iii), we notice that due to the fact that the components of the currying are bijections, it is enough to prove that (dropping indices from the currying)

$$\begin{aligned}
&\zeta^{-1}\big(\bullet_{V,W,Y} \circ (\bullet_{W,X,Y} \otimes \mathrm{id}_{\mathrm{hom}(V,W)})\big) \\
&= \zeta^{-1}\big(\bullet_{V,X,Y} \circ (\mathrm{id}_{\mathrm{hom}(X,Y)} \otimes \bullet_{V,W,X}) \circ \Phi_{\mathrm{hom}(X,Y),\mathrm{hom}(W,X),\mathrm{hom}(V,W)}\big) \ .
\end{aligned} \tag{2.2.63}$$

This equality is shown by applying item (i) and (ii) and using the 3-cocycle condition (2.1.98c) and the H-equivariance of the internal composition (see B.3). $\square$

2.2.8 The braiding

Using the (invertible) R-matrix $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ of H, one can define a natural isomorphism $\tau : \otimes \Rightarrow \otimes^{\mathrm{op}}$ by setting

$$\tau_{\rho_V,\rho_W} : \rho_V \otimes \rho_W \Rightarrow \rho_W \otimes \rho_V \ , \tag{2.2.64}$$

with single component

$$\tau_{V,W} := (\rho_W \otimes \rho_V)(R_{21}) \circ \sigma_{V,W} \ , \tag{2.2.65}$$

where σ is the braiding in $\mathscr{M}$. It follows from (2.1.103a) that τ_{ρ_V,ρ_W} is an $[H,\mathscr{M}]$-morphism. We have (suppressing σ)

$$\begin{aligned}
\rho_W \otimes \rho_V(h) \circ \tau_{V,W} &= (\rho_W \otimes \rho_V)(\Delta(h)) \circ (\rho_V \otimes \rho_W)(R_{21}) \\
&= (\rho_V \otimes \rho_W)(\Delta^{\mathrm{op}}(h) \cdot R_{21}) \\
&= (\rho_V \otimes \rho_W)(R_{21} \cdot \Delta(h)) \\
&= (\rho_V \otimes \rho_W)(R_{21}) \circ (\rho_V \otimes \rho_W)(\Delta(h)) \\
&= \tau_{V,W} \circ \rho_V \otimes \rho_W(h) \ ,
\end{aligned} \tag{2.2.66}$$

where the third step follows from the equalities $\Delta^{\mathrm{op}}(h) \cdot R_{21} = [\Delta(h) \cdot R]_{21} = [R \cdot \Delta^{\mathrm{op}}(h)]_{21} = R_{21} \cdot \Delta(h)$. As a direct consequence of (2.1.103b), (2.1.103c) the components of the natural isomorphism τ satisfy the hexagon relations. Using (2.1.103c) (and suppressing σ)

$$\begin{aligned}
\tau_{V\otimes W,Z} &= (\rho_V \otimes \rho_W) \otimes \rho_Z(R_{21}) \\
&= ((\rho_V \otimes \rho_W) \otimes \rho_Z)[(\Delta \otimes \mathrm{id})(R)]_{231} \\
&= ((\rho_V \otimes \rho_W) \otimes \rho_Z)\big[\phi_{312}\, R_{13}\, \phi_{132}^{-1}\, R_{23}\, \phi_{123}\big]_{231} \\
&= \Phi_{Z,V,W} \circ (\tau_{V,Z} \otimes \mathrm{id}_W) \circ \Phi_{V,Z,W}^{-1} \circ (\mathrm{id}_V \otimes \tau_{W,Z}) \circ \Phi_{V,W,Z} \ ,
\end{aligned} \tag{2.2.67}$$

where in the last step we have used the functoriality of representations. By a similar calculation using (2.1.103b) (see B.4)

$$\tau_{V,W\otimes Z} = \Phi^{-1}_{W,Z,V} \circ (\mathrm{id}_W \otimes \tau_{V,Z}) \circ \Phi_{W,V,Z} \circ (\tau_{V,W} \otimes \mathrm{id}_Z) \circ \Phi^{-1}_{V,W,Z} \ . \tag{2.2.68}$$

In summary,

Proposition 2.2.14. *For any quasitriangular quasi-Hopf algebra H the category $[H, \mathscr{M}]$ of left H-modules is a braided monoidal category with braiding given by (2.2.65).*

Remark 2.2.15. In general the $[H, \mathscr{M}]$-morphism $\tau_{W,V} \circ \tau_{V,W} : \rho_V \otimes \rho_W \Rightarrow \rho_V \otimes \rho_W$ does not coincide with the identity morphism $\mathrm{id}_{V\otimes W}$, hence the braided closed monoidal category $[H, \mathscr{M}]$ is not symmetric: The inverse of $\tau_{V,W}$ is given by the braiding $\tau'_{W,V}$ induced by the second R-matrix $R' := R^{-1}_{21}$, cf. Remark 2.1.37. However for a triangular quasi-Hopf algebra there is the additional property (2.1.104), which implies that $R = R'$ and hence $\tau_{W,V} \circ \tau_{V,W} = \mathrm{id}_{V\otimes W}$. Thus the representation category $[H, \mathscr{M}]$ of a triangular quasi-Hopf algebra H is a symmetric monoidal category.

2.2.9 Cochain twisting the braiding

For a quasitriangular quasi-Hopf algebra H, it follows from Theorem 2.1.43 that H_F is a quasitriangular quasi-Hopf algebra with R-matrix R_F. Proposition 2.2.14 then implies that $[H_F, \mathscr{M}]$ is also a braided monoidal category.

Theorem 2.2.16. *For any quasitriangular quasi-Hopf algebra H and any cochain twist $F \in H \otimes H$, the equivalence of monoidal categories in Theorem 2.2.8 is an equivalence between the braided monoidal categories $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$.*

Proof. Denoting the braiding in $[H, \mathscr{M}]$ by τ and that in $[H_F, \mathscr{M}]$ by τ^F, it is required to show that the diagram

$$
\begin{tikzcd}
\mathcal{F}(\rho_V)\otimes_F \mathcal{F}(\rho_W) \arrow[r, Rightarrow, "\tau^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W)}"] \arrow[d, Rightarrow, "\varphi_{\rho_V,\rho_W}"'] & \mathcal{F}(\rho_W)\otimes_F \mathcal{F}(\rho_V) \arrow[d, Rightarrow, "\varphi_{\rho_W,\rho_V}"] \\
\mathcal{F}(\rho_V\otimes\rho_W) \arrow[r, Rightarrow, "\mathcal{F}(\tau_{\rho_V,\rho_W})"'] & \mathcal{F}(\rho_W\otimes\rho_V)
\end{tikzcd}
\tag{2.2.69}
$$

commutes for any two objects ρ_V, ρ_W in $[H,\mathscr{M}]$. This is a direct consequence of the definition of the twisted R-matrix (2.1.112), together with (2.2.65) and (2.2.25): we have (suppressing σ)

$$
\begin{aligned}
\varphi_{W,V}\circ\tau^F_{\mathcal{F}(V),\mathcal{F}(W)} &= (\rho_W\otimes\rho_V)(F^{-1})\circ(\rho_W\otimes\rho_V)(R_{F21})\\
&= (\rho_W\otimes\rho_V)(F^{-1}\cdot R_{F21})\\
&= (\rho_W\otimes\rho_V)(R_{21}\cdot F_{21}^{-1})\\
&= (\rho_W\otimes\rho_V)(R_{21})\circ(\rho_V\otimes\rho_W)(F^{-1})\\
&= \mathcal{F}(\tau_{V,W})\circ\varphi_{V,W}\ .
\end{aligned}
\tag{2.2.70}
$$

□

2.2.10 Algebras in $[H,\mathscr{M}]$

Definition 2.2.17 (Algebra)**.** Let H be a quasi-Hopf algebra. An *algebra* in $[H,\mathscr{M}]$ is a monoid object (ρ_A,μ_A,η_A) in the monoidal category $[H,\mathscr{M}]$ (c.f. Definition 2.1.8). Here $\mu_A:\rho_A\otimes\rho_A\Rightarrow\rho_A$ and $\eta_A:\rho_I\Rightarrow\rho_A$ are the multiplication and unit $[H,\mathscr{M}]$-morphisms.

Definition 2.2.18 (Category of algebras)**.** Let H be a quasi-Hopf algebra. The collection of algebra objects in $[H,\mathscr{M}]$ together with $[H,\mathscr{M}]$-morphisms $f:\rho_A\Rightarrow\rho_B$ which preserve the product μ_A and unit η_A, i.e. such that

$$
f\circ\mu_A=\mu_B\circ(f\otimes_{\rho_I}f)\ ,\ \ f\circ\eta_A=\eta_B\circ\mathrm{id}_{\rho_I}\ , \tag{2.2.71}
$$

constitute a subcategory of $[H,\mathscr{M}]$. This subcategory is equal to the pair of comma categories $(\otimes_{\rho_I} \Rightarrow \mathrm{id}_{[H,\mathscr{M}]})$ and $(\mathrm{id}_{[H,\mathscr{M}]} \Rightarrow \mathrm{id}_{[H,\mathscr{M}]})$ whose objects are pairs of triples $(\rho_A \times \rho_A, \mu_A, \rho_A)$ and (ρ_I, η_A, ρ_A) with (ρ_A, μ_A, η_A) a monoid object in $[H,\mathscr{M}]$ and whose morphisms are pairs of tuples of morphisms $(f \times f, f)$ and $(\mathrm{id}_{\rho_I}, f)$ satisfying (2.2.71). We shall denote by

$$H\text{-Alg} \ , \tag{2.2.72}$$

the *category of algebras* in $[H,\mathscr{M}]$. And with an abuse of notation denote objects in H-Alg by the corresponding objects in $[H,\mathscr{M}]$.

We note that since the associator in $[H,\mathscr{M}]$ is not trivial an algebra ρ_A in H-Alg is in general not an associative algebra, but only *weakly associative*, i.e. associative up to the associator in H.

Before considering the important example 2.1.9 in the context of $[H,\mathscr{M}]$ we collect some useful properties involving the element β of the quasi-antipode.

Lemma 2.2.19. *Let H be a quasi-Hopf algebra and ρ_V any object in $[H,\mathscr{M}]$. Noting that $\rho_V(\beta) \in \mathrm{end}(V)$, we have*

$$\rho_{\mathrm{end}(V)}(h)\big(\rho_V(\beta)\big) = \rho_V(\epsilon(h)\beta) \tag{2.2.73a}$$

$$\mathrm{ev}_{V,V}(\rho_V(\beta) \otimes -) = \rho_V(1_H) \ , \tag{2.2.73b}$$

for any $h \in H$.

Proof. By the functoriality of representations and property (2.1.100b) of the quasi-antipode

$$\rho_{\mathrm{end}(V)}(h)\big(\rho_V(\beta)\big) = \rho_V(h_{(1)}) \circ \rho_V(\beta) \circ \rho_V(S(h_{(2)})) = \rho_V(\epsilon(h)\beta) \ , \tag{2.2.74}$$

and again by the functoriality of representations and property (2.1.100c) of the quasi-antipode

$$\mathrm{ev}_{V,V}(\rho_V(\beta) \otimes -) = \rho_V(\phi^{(1)}) \circ \rho_V(\beta) \circ \rho_V\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\big) = \rho_V(1_H) \ . \tag{2.2.75}$$

□

Example 2.2.20. Given an object ρ_V in $[H,\mathscr{M}]$ its internal endomorphisms $\mathrm{end}(\rho_V) := \mathrm{hom}(\rho_V,\rho_V)$ is an object in $[H,\mathscr{M}]$. Since by Proposition 2.2.13 (iii) the internal composition is weakly associative (i.e. associative up to the associator as in the first diagram of Definition 2.1.8) it defines a weakly associative product on the internal endomorphisms

$$\mu_{\mathrm{end}(V)} := \bullet_{V,V,V} : \mathrm{end}(\rho_V)\otimes\mathrm{end}(\rho_V) \Longrightarrow \mathrm{end}(\rho_V)\ . \tag{2.2.76}$$

Furthermore due to the currying ζ in (2.2.36) we can assign to the $[H,\mathscr{M}]$-morphism $\lambda_V : \rho_I\otimes\rho_V \Rightarrow \rho_V$ the $[H,\mathscr{M}]$-morphism

$$\eta_{\mathrm{end}(V)} := \zeta_{I,V,V}(\lambda_V) : \rho_I \Longrightarrow \mathrm{end}(\rho_V)\ . \tag{2.2.77}$$

Explicitly, evaluating the single component of this morphism on $1\in I$ we find

$$\eta_{\mathrm{end}(V)}(1) = \rho_V(\beta)\ . \tag{2.2.78}$$

Using the properties in Lemma 2.2.19, we have for any $L\in\mathrm{end}(V)$

$$\begin{aligned}
\rho_V(\beta)\bullet L &= \zeta(\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi)\big((\rho_V(\beta)\otimes L)\big)\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi\circ\big((\rho_{\mathrm{end}(V)}\otimes\rho_{\mathrm{end}(V)})\otimes\rho_V\big)\\
&\big((\Delta\otimes 1\otimes S)(\phi^{-1})_{1\hat{2}4}\beta_{\hat{3}}\big)(\rho_V(\beta)\otimes L\otimes -)\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\big((1_{\mathrm{End}(V)}\otimes\rho_{\mathrm{end}(V)})\otimes\rho_V\big)\\
&\big((1\otimes 1\otimes S)(\phi^{-1})_{1\hat{2}4}\beta_{\hat{3}}\big)(\rho_V(\beta)\otimes L\otimes -)\\
&= \mathrm{ev}\circ(\rho_{\mathrm{end}(V)}\otimes\rho_V)\big((1\otimes 1\otimes S)(\phi^{-1})_{1\hat{2}4}\beta_{\hat{3}}\big)(L\otimes -)\\
&= \zeta_{\mathrm{end}(V),V,V}(\mathrm{ev}_{V,V})(L)\\
&= \zeta_{\mathrm{end}(V),V,V}\circ\zeta^{-1}_{\mathrm{end}(V),V,V}(\mathrm{id}_{\mathrm{end}(V)})(L)\\
&= L\ ,
\end{aligned}\tag{2.2.79}$$

where in the third and fourth equality we have used (2.2.73a) and property (2.1.98a) and (2.1.99) respectively. In the fifth equality we have used (2.2.73b) and in the sixth equality follows from (2.2.38) and (2.2.53). By a similar calculation we have

$$L \bullet \rho_V(\beta) = L \ . \tag{2.2.80}$$

Hence

$$\eta_{\mathrm{end}(V)}(1) = 1_{\mathrm{end}(V)} \ , \tag{2.2.81}$$

and $\big(\mathrm{end}(\rho_V), \mu_{\mathrm{end}(V)}, \eta_{\mathrm{end}(V)}\big)$ is an algebra in $[H, \mathscr{M}]$.

Remark 2.2.21. Given an object ρ_V in $[H, \mathscr{M}]$, the algebra $\mathrm{end}(\rho_V)$ in $[H, \mathscr{M}]$ describes the (nonassociative) algebra of linear operators on V. A *representation* of an object ρ_A in H-Alg on V is then defined to be an H-Alg-morphism $\pi_A : \rho_A \Rightarrow \mathrm{end}(\rho_V)$.

In the following let us fix a quasitriangular quasi-Hopf algebra H and denote the R-matrix by $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$.

Definition 2.2.22 (Braided commutative algebra)**.** Let H be a quasitriangular quasi-Hopf algebra. An algebra ρ_A in $[H, \mathscr{M}]$ is called *braided commutative* if it is a commutative algebra in $[H, \mathscr{M}]$ (cf. Definition 2.1.15). We denote the full subcategory of H-Alg of braided commutative algebras in $[H, \mathscr{M}]$ by H-$\mathrm{Alg}^{\mathrm{com}}$.

Remark 2.2.23. Recall that the braiding τ' which is determined by the second R-matrix $R' := R_{21}^{-1}$ (cf. Remark 2.1.37) is related to the original braiding τ by $\tau'_{V,W} = \tau^{-1}_{W,V}$. As a consequence, the commutative diagram (2.1.43) is equivalent to the same diagram with τ replaced by τ'. In other words, braided commutative algebras in $[H, \mathscr{M}]$ are braided commutative with respect to both quasitriangular structures R and R' on H.

2.2.11 Cochain twisting algebras

Given any cochain twist $F \in H \otimes H$, recall that there is a monoidal functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$. Thus given any algebra ρ_A in $[H, \mathscr{M}]$ $\mathcal{F}(\rho_A)$ is an object in $[H_F, \mathscr{M}]$. For this object define the $[H_F, \mathscr{M}]$-morphisms $\mu_{A_F} : \mathcal{F}(\rho_A) \otimes_F \mathcal{F}(\rho_A) \Rightarrow \mathcal{F}(\rho_A)$ and $\eta_{A_F} : \rho_{I_F} \Rightarrow \mathcal{F}(\rho_A)$ via the coherence maps (2.2.24) and the diagrams

$$\begin{tikzcd}
\mathcal{F}(\rho_A) \otimes_F \mathcal{F}(\rho_A) \arrow[r, Rightarrow, "\mu_{\rho_{A_F}}"] \arrow[d, Rightarrow, "\varphi_{\rho_A,\rho_A}"'] & \mathcal{F}(\rho_A) \\
\mathcal{F}(\rho_A \otimes \rho_A) \arrow[ur, Rightarrow, "\mathcal{F}(\mu_{\rho_A})"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho_{I_F} \arrow[r, Rightarrow, "\eta_{\rho_{A_F}}"] \arrow[d, Rightarrow, "\psi"'] & \mathcal{F}(\rho_A) \\
\mathcal{F}(\rho_I) \arrow[ur, Rightarrow, "\mathcal{F}(\eta_{\rho_A})"'] &
\end{tikzcd}
\tag{2.2.82}$$

in $[H, \mathscr{M}]$. That is

$$\mu_{A_F} = \mathcal{F}(\mu_A) \circ \varphi_{A,A} \ , \quad \eta_{A_F} = \mathcal{F}(\eta_A) \circ \psi \ . \tag{2.2.83}$$

It is easy to see that $\mathcal{F}(\rho_A)$, together with the $[H_F, \mathscr{M}]$-morphisms μ_{A_F} and η_{A_F}, is an algebra in $[H_F, \mathscr{M}]$. Denote this algebra also by $\mathcal{F}(\rho_A)(*) = A_F$. For any H-Alg-morphism $f : \rho_A \Rightarrow \rho_B$ the $[H_F, \mathscr{M}]$-morphism $\mathcal{F}(f) : \mathcal{F}(\rho_A) \Rightarrow \mathcal{F}(\rho_B)$ is also an H_F–Alg-morphism with single component (denoted by the same symbol) the k-linear map $\mathcal{F}(f) : A_F \to B_F$. Thus one obtains a functor $\mathcal{F} : H\text{-Alg} \to H_F\text{–Alg}$, which is invertible by using the cochain twist F^{-1} based on H_F (cf. Remark 2.1.41). In summary,

Proposition 2.2.24. *If H is a quasi-Hopf algebra and $F \in H \otimes H$ is any cochain twist based on H, then the categories H-Alg and H_F–Alg are equivalent.*

The braided symmetry property is preserved under cochain twisting.

Proposition 2.2.25. *Let H be a quasitriangular quasi-Hopf algebra and $F \in H \otimes H$ any cochain twist based on H. Then the equivalence between the categories H-Alg and H_F–Alg of Proposition 2.2.24 restricts to an equivalence between the full subcategories $H\text{-Alg}^{\mathrm{com}}$ and $H_F\text{–Alg}^{\mathrm{com}}$.*

Proof. This is an immediate consequence of the definition of the twisted R-matrix (2.1.112) and the twisted algebra product (2.2.82): For any object ρ_A in $H\text{-Alg}^{\mathrm{com}}$

we have

$$\begin{aligned}\mu_F &= \mu \circ (\rho_A \otimes \rho_A)(F^{-1}) \\ &= \mu \circ (\rho_A \otimes \rho_A)(R_{21}\, F_{21}^{-1}) \circ \sigma \\ &= \mu \circ (\rho_A \otimes \rho_A)(F^{-1} \cdot F\, R_{21}\, F_{21}^{-1}) \circ \sigma \\ &= \mu_F \circ (\rho_A \otimes \rho_A)(R_{F21}) \circ \sigma \\ &= \mu_F \circ \tau_{A,A} \ . \end{aligned} \tag{2.2.84}$$

Hence $\mathcal{F}(\rho_A)$ is an object in H_F–$\mathrm{Alg}^{\mathrm{com}}$. $\square$

Example 2.2.26. If H is any cocommutative quasi-Hopf algebra with trivial R-matrix $R = 1 \otimes 1$ then commutative algebras ρ_A in $[H, \mathscr{M}]$ are braided commutative. Such examples arise in ordinary differential geometry, see Chapter 4. From Proposition 2.2.25 any cochain twisting of such examples satisfies the braided commutativity condition. This will be our main source of examples.

2.2.12 The internal tensor product

By Proposition 2.2.14 and Theorem 2.2.11 the representation category $[H, \mathscr{M}]$ of a quasitriangular quasi-Hopf algebra H is a braided closed monoidal category. So by Proposition 2.1.7 there is a tensor product morphism for the internal hom-objects

$$\circledast_{V,W,X,Y} : \hom(\rho_V, \rho_W) \otimes \hom(\rho_X, \rho_Y) \Longrightarrow \hom(\rho_V \otimes \rho_X, \rho_W \otimes \rho_Y) \ , \tag{2.2.85}$$

for all objects $\rho_V, \rho_W, \rho_X, \rho_Y$ in $[H, \mathscr{M}]$ given by (dropping indices)

$$\circledast := \zeta\Big((\mathrm{ev} \otimes \mathrm{ev}) \circ \Phi^{-1} \circ (\mathrm{id} \otimes \Phi) \circ (\mathrm{id} \otimes (\tau \otimes \mathrm{id})) \circ (\mathrm{id} \otimes \Phi^{-1}) \circ \Phi \Big) \tag{2.2.86}$$

The most general formula for the internal tensor product morphism is impractically lengthy. When one of the internal homomorphisms is the unit of an internal endomorphism algebra, i.e. for internal tensor products of the following form $L \circledast_{V,W,X,X} 1_{\mathrm{end}(X)}$ or $1_{\mathrm{end}(V)} \circledast_{V,V,X,Y} L'$, for any $L \in \hom(V, W)$ and $L' \in \hom(X, Y)$, the formulae are considerably simpler. Recall Example 2.2.20, for any object ρ_V in $[H, \mathscr{M}]$ there is the internal endomorphism algebra $\mathrm{end}(\rho_V) = \mathrm{hom}(\rho_V, \rho_V)$ with product given by $\mu_{\mathrm{end}(V)} = \bullet_{V,V,V}$ and unit $1_{\mathrm{end}(V)} = \rho_V(\beta)$.

We now explicitly compute the internal homomorphisms $L \circledast_{V,W,X,X} 1_{\mathrm{end}(X)}$ and $1_{\mathrm{end}(V)} \circledast_{V,V,X,Y} L'$, for any $L \in \mathrm{hom}(V, W)$ and $L' \in \mathrm{hom}(X, Y)$, from which we later derive properties of $L \circledast_{V,W,X,Y} L'$.

Using (2.1.28) and the fact that the identity element $1_{\mathrm{end}(X)}$ is H-invariant (cf. (2.2.73a) recalling that $1_{\mathrm{end}(X)} = \rho_X(\beta)$ from (2.2.81), (2.2.78)), we obtain (dropping indices on the currying and evaluation)

$$
\begin{aligned}
&\zeta^{-1}(\circledast)(L \otimes 1_{\mathrm{end}(X)} \otimes - \otimes -)\\
&= \mathrm{ev}\big(\rho_{\mathrm{hom}(V,W)}(\phi^{(-1)})(L) \otimes \rho_V(\phi^{(-2)})\big) \otimes \mathrm{ev}\big(1_{\mathrm{end}(X)} \otimes \rho_X(\phi^{(-3)})\big)\\
&= \mathrm{ev}\big(\rho_{\mathrm{hom}(V,W)}(\phi^{(-1)})(L) \otimes \rho_V(\phi^{(-2)})\big) \otimes \rho_X(\phi^{(-3)})\\
&= (\mathrm{ev} \otimes \mathrm{id}_X) \circ \big((\rho_{\mathrm{hom}(V,W)} \otimes \rho_V) \otimes \rho_X\big)(\phi^{-1})(L \otimes - \otimes -) \ ,
\end{aligned} \tag{2.2.87}
$$

where in the second equality we have used (2.2.73b).

Lemma 2.2.27. *Let H be a quasi-Hopf algebra and $F \in H \otimes H$ any cochain twist based on H. If f is an $[H, \mathscr{M}]$-morphism, then*

$$\zeta(f) = f \circ \zeta(\mathrm{id}) \ . \tag{2.2.88}$$

Proof. (2.2.88) follows directly from (2.2.38). □

We therefore have

$$
\begin{aligned}
&L \circledast 1_{\mathrm{end}(X)} =\\
&(\mathrm{ev} \otimes \mathrm{id}_X) \circ \big((\rho_{\mathrm{hom}(V,W)} \otimes \rho_V) \otimes \rho_X\big)(\phi^{-1}) \circ \zeta(\mathrm{id})(L \otimes \mathrm{id}_V \otimes \mathrm{id}_X) \ .
\end{aligned} \tag{2.2.89}
$$

By a similar calculation

$$
\begin{aligned}
&\zeta^{-1}(\circledbullet)(1_{\mathrm{end}(V)} \otimes L' \otimes - \otimes -) \\
&= \rho_V(\widetilde{\phi}^{(1)}\, R^{(2)}\, \phi^{(-2)}) \otimes \mathrm{ev}\big(\rho_{\mathrm{hom}(X,Y)}(\widetilde{\phi}^{(2)}\, R^{(1)}\, \phi^{(-1)})(L') \otimes \rho_X(\widetilde{\phi}^{(3)}\, \phi^{(-3)})\big) \\
&= (\mathrm{id}_V \otimes \mathrm{ev}) \circ (\rho_V \otimes (\rho_{\mathrm{hom}(X,Y)} \otimes \rho_X))(\phi \cdot R_{21} \cdot \phi_{213}^{-1})(- \otimes L' \otimes -) \ ,
\end{aligned} \tag{2.2.90}
$$

and therefore

$$
\begin{aligned}
&1_{\mathrm{end}(V)} \circledbullet L' = \\
&(\mathrm{id}_V \otimes \mathrm{ev}) \circ (\rho_V \otimes (\rho_{\mathrm{hom}(X,Y)} \otimes \rho_X))(\phi \cdot R_{21} \cdot \phi_{213}^{-1}) \circ \zeta(\mathrm{id})(\mathrm{id}_V \otimes L' \otimes \mathrm{id}_X) \ .
\end{aligned} \tag{2.2.91}
$$

As a consequence of (2.2.89) (or (2.2.91)) we have

$$
1_{\mathrm{end}(V)} \circledbullet_{V,V,X,X} 1_{\mathrm{end}(X)} = (\rho_V \otimes \rho_X)(\Delta(\beta)) = \rho_V \otimes \rho_X(\beta) = 1_{\mathrm{end}(V\otimes X)} \ . \tag{2.2.92}
$$

where in the last step we have used (cf. (2.2.81), (2.2.78), (2.2.17))

$$
1_{\mathrm{end}(V\otimes X)} = \eta_{\mathrm{end}(V\otimes X)}(1) = \rho_{V\otimes X}(\beta) = \rho_V \otimes \rho_X(\beta) \ . \tag{2.2.93}
$$

We now study compatibility properties between the internal tensor product $\circledbullet$ and the composition $\bullet$. We begin by clarifying these properties for four special cases.

Lemma 2.2.28. *For any $L \in \mathrm{hom}(V,W)$, $L' \in \mathrm{hom}(X,Y)$, $K \in \mathrm{hom}(W,X)$ and $K' \in \mathrm{hom}(Y,Z)$ one has*

$$
\begin{aligned}
&(L \bullet 1_{\mathrm{end}(V)}) \circledbullet_{V,W,X,Y} (1_{\mathrm{end}(Y)} \bullet L') = \\
&\qquad \big(L \circledbullet_{V,W,Y,Y} 1_{\mathrm{end}(Y)}\big) \bullet_{V\otimes X, V\otimes Y, W\otimes Y} \big(1_{\mathrm{end}(V)} \circledbullet_{V,V,X,Y} L'\big) \ ,
\end{aligned} \tag{2.2.94a}
$$

$$
\begin{aligned}
&\big(K \bullet_{V,W,X} L\big) \circledbullet_{V,X,Y,Y} 1_{\mathrm{end}(Y)} = \\
&\qquad \big(K \circledbullet_{W,X,Y,Y} 1_{\mathrm{end}(Y)}\big) \bullet_{V\otimes Y, W\otimes Y, X\otimes Y} \big(L \circledbullet_{V,W,Y,Y} 1_{\mathrm{end}(Y)}\big) \ ,
\end{aligned} \tag{2.2.94b}
$$

$$1_{\mathrm{end}(V)} \circledbullet_{V,V,X,Z} \left(K' \bullet_{X,Y,Z} L'\right) = \left(1_{\mathrm{end}(V)} \circledbullet_{V,V,Y,Z} K'\right) \bullet_{V\otimes X, V\otimes Y, V\otimes Z} \left(1_{\mathrm{end}(V)} \circledbullet_{V,V,X,Y} L'\right) \,, \quad (2.2.94c)$$

$$\left(1_{\mathrm{end}(W)} \bullet \left(R^{(2)} \triangleright_{\mathrm{hom}(V,W)} L\right)\right) \circledbullet_{V,W,X,Y} \left(\left(R^{(1)} \triangleright_{\mathrm{hom}(X,Y)} L'\right) \bullet 1_{\mathrm{end}(X)}\right) = \left(1_{\mathrm{end}(W)} \circledbullet_{W,W,X,Y} L'\right) \bullet_{V\otimes X, W\otimes X, W\otimes Y} \left(L \circledbullet_{V,W,X,X} 1_{\mathrm{end}(X)}\right) \,. \quad (2.2.94d)$$

Proof. By Proposition 2.2.13 (i) and bijectivity of the currying maps, It is enough to prove that the equalities hold after evaluation on generic elements. The evaluation of the left-hand side of the equality (2.2.94a) is easily computed from (2.1.28), while the evaluation of the right-hand side can be simplified by first using Proposition 2.2.13 (ii) and then (2.2.90),(2.2.87). It is then easy to check that both expressions agree.

The equality (2.2.94b) is easily proven by first evaluating both sides and then using Proposition 2.2.13 (ii), (2.2.87) and the 3-cocycle condition (2.1.98c) to simplify the expressions.

The equality (2.2.94c) is slightly more complicated to prove. We again evaluate both sides and use Proposition 2.2.13 (ii) together with (2.2.90) to simplify the expressions. The problem then reduces to proving that

$$\big[(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)(\phi)\big]_{3124} R_{13} \big[(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)(\phi^{-1})\big]_{1324} \cdot \phi_{324} R_{23} \phi^{-1}_{234} \big[(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)(\phi)\big]_{1234} \quad (2.2.95a)$$

is equal to

$$\phi_{124} \big[(\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_H)(\phi)\big]_{3124} \big[(\Delta \otimes \mathrm{id}_H)(R)\big]_{123} \big[(\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi^{-1})\big]_{1234} \,. \quad (2.2.95b)$$

Multiplying the expressions in (2.2.95) from the left by $\big[(\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_H)(\phi^{-1})\big]_{3124} \phi^{-1}_{124}$ and from the right by $\big[(\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi)\big]_{1234}$, the expression (2.2.95b) becomes $\big[(\Delta \otimes \mathrm{id}_H)(R)\big]_{123}$. Simplifying the expression (2.2.95a) by applying the 3-cocycle condition (2.1.98c) three times, the R-matrix property (2.1.103a) twice and then the R-matrix property (2.1.103c) it also becomes $\big[(\Delta\otimes\mathrm{id}_H)(R)\big]_{123}$. This proves (2.2.94c).

To prove the equality (2.2.94d) we again evaluate both sides and use Proposition 2.2.13 (ii), (2.2.94a) and (2.2.90),(2.2.87) to simplify the expressions. The problem then reduces to proving that

$$\begin{aligned}&\big[(\Delta\otimes\mathrm{id}_H\otimes\mathrm{id}_H)(\phi)\big]_{2314}\,\big[(\mathrm{id}_H\otimes\Delta)(R)\big]_{123}\\ &\qquad\cdot\,\big[(\mathrm{id}_H\otimes\Delta\otimes\mathrm{id}_H)(\phi^{-1})\big]_{1234}\,\phi^{-1}_{234}\,\big[(\mathrm{id}_H\otimes\mathrm{id}_H\otimes\Delta)(\phi)\big]_{1234}\end{aligned}\tag{2.2.96a}$$

is equal to

$$\big[(\mathrm{id}_H\otimes\mathrm{id}_H\otimes\Delta)(\phi^{-1})\big]_{2314}\,\phi_{314}\,R_{13}\,\phi^{-1}_{134}\,\big[(\mathrm{id}_H\otimes\mathrm{id}_H\otimes\Delta)(\phi)\big]_{2134}\,R_{12}\ .\tag{2.2.96b}$$

This follows from the 3-cocycle condition (2.1.98c) and the R-matrix properties (2.1.103a), (2.1.103b). □

To simplify the notation throughout the rest of this section we shall drop all labels on $\circledast$, $\bullet$, Φ.

With this preparation we have

Proposition 2.2.29. *Let H be a quasitriangular quasi-Hopf algebra. Then the internal tensor product $\circledast$ satisfies the braided composition property, i.e.*

$$\bullet\circ(\circledast\otimes\circledast)=\circledast\circ(\bullet\otimes\bullet)\circ\Phi^{-1}\circ(\mathrm{id}\otimes\Phi)\circ(\mathrm{id}\otimes(\tau\otimes\mathrm{id}))\circ(\mathrm{id}\otimes\Phi^{-1})\circ\Phi\ .\tag{2.2.97}$$

Proof. This is a direct calculation using Lemma 2.2.28 and weak associativity of the internal composition $\bullet$, cf. Proposition 2.2.13 (iii). □

It remains to prove that the internal tensor product $\circledast$ is weakly associative.

Proposition 2.2.30. *Let H be a quasitriangular quasi-Hopf algebra. Then the* *internal tensor product* $\circledast$ *is weakly associative, i.e.*

$$\Phi \circ (\,\cdot\,) \circ \Phi^{-1} \circ \circledast \circ (\circledast \otimes \mathrm{id}) = \circledast \circ (\mathrm{id} \otimes \circledast) \ . \tag{2.2.98}$$

Proof. On the left hand side of (2.2.98) we obtain

$$\begin{aligned}
&\Phi \circ \big((L \circledast L'\,) \circledast L''\,\big) \circ \Phi^{-1} \\
&\qquad = \Phi \circ \Big(\big(((L \circledast 1) \bullet (1 \circledast L'\,)) \circledast 1\big) \bullet \big((1 \circledast 1) \circledast L''\,\big)\Big) \circ \Phi^{-1} \\
&\qquad = \Phi \circ \Big(\Big(\big((L \circledast 1) \circledast 1\big) \bullet \big((1 \circledast L'\,) \circledast 1\big)\Big) \bullet \big((1 \circledast 1) \circledast L''\,\big)\Big) \circ \Phi^{-1} \\
&\qquad = \Big(\big(\Phi \circ \big((L \circledast 1) \circledast 1\big) \circ \Phi^{-1}\big) \bullet \big(\Phi \circ \big((1 \circledast L'\,) \circledast 1\big) \circ \Phi^{-1}\big)\Big) \bullet \\
&\qquad\qquad \big(\Phi \circ \big((1 \circledast 1) \circledast L''\,\big) \circ \Phi^{-1}\big) \ .
\end{aligned} \tag{2.2.99}$$

In the first and second equalities we used equation (2.2.92) and Lemma 2.2.28. The third equality follows from the H-equivariance of $\bullet$ which enables one to introduce $\mathrm{id} = \Phi^{-1} \circ \Phi$ and split it on either side of $\bullet$. By a straightforward computation using (2.2.90), (2.2.87) one checks the equalities

$$\Phi \circ \big((L \circledast 1) \circledast 1\big) \circ \Phi^{-1} = L \circledast \big(1 \circledast 1\big) \ , \tag{2.2.100a}$$

$$\Phi \circ \big((1 \circledast L'\,) \circledast 1\big) \circ \Phi^{-1} = 1 \circledast \big(L' \circledast 1\big) \ , \tag{2.2.100b}$$

$$\Phi \circ \big((1 \circledast 1) \circledast L''\,\big) \circ \Phi^{-1} = 1 \circledast \big(1 \circledast L''\,\big) \ , \tag{2.2.100c}$$

which together with (2.2.99) and weak associativity of the composition morphisms $\bullet$ (cf. Proposition 2.2.13 (iii)) implies the equation (2.2.98). □

2.2.13 The internal commutator

Proposition 2.2.31. *Let H be a triangular quasi-Hopf algebra. The internal commutator in the category $\mathscr{M}$ (cf. Definition 2.1.11) restricts to an internal commutator satisfying the braided antisymmetry, Jacobi and biderivation properties interpreted in the category $[H, \mathscr{M}]$.*

Proof. First we note that the target of the commutator is an H-module since the commutator is an H-module morphism. The braided antisymmetry follows from the same calculation as in Definition 2.1.11 noting that from the triangularity of the R-matrix we have $\tau^{-1} = \tau$. The proofs of the Jacobi identity and biderivation properties involve standard manipulations using the weak associativity of the internal composition (2.2.60) and standard properties of the triangular R-matrix. □

Remark 2.2.32. The biderivation property above holds for an arbitrary quasitriangular quasi-Hopf algebra.

Corollary 2.2.33. *Let H be a triangular quasi-Hopf algebra and ρ_V any object in $[H, \mathscr{M}]$. Then the $[H, \mathscr{M}]$-object given by the internal endomorphisms* $\mathrm{end}(\rho_V)$*, together with the internal commutator $[\,\cdot\,,\,\cdot\,]$ given in (2.1.39) in the context of $[H, \mathscr{M}]$, is a Lie algebra in $[H, \mathscr{M}]$.*

2.2.14 Cochain twisting the map-like structures

The evaluation ev, internal composition $\bullet$, internal tensor product $\circledast$ and internal commutator $[\,\cdot\,,\,\cdot\,]$ described in the previous subsections are the appropriate structures with which to use internal homomorphisms correctly as map-like objects in $[H, \mathscr{M}]$. Although in the category $[H, \mathscr{M}]$ internal homomorphisms are k-linear maps they do not give the correct behaviour under the usual structures of evaluation, composition and tensor product.

The results in the following lemma are very useful for proving properties of the cochain twisting of map-like structures for internal homomorphisms.

Lemma 2.2.34. *Let H be a quasi-Hopf algebra and $F \in H \otimes H$ any cochain twist based on H. The currying bijection ζ_F for internal* hom*-objects in $[H_F, \mathscr{M}]$ can be written in terms of the currying bijection ζ for internal* hom*-objects in $[H, \mathscr{M}]$ by*

$$\zeta_F(\mathrm{id}) = \varphi^{-1} \circ (\gamma^{-1} \otimes_F \mathrm{id}) \circ \mathcal{F}(\zeta(\mathrm{id})) \ , \tag{2.2.101}$$

where $\mathcal{F}$ is the equivalence between the categories $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$ (cf. Theorem 2.2.5).

Proof. Starting with (2.2.38) in the category $[H_F, \mathscr{M}]$ we notice that the properties of the quasi-antipode, coproduct and associator in H_F are such that:

$$
\begin{aligned}
&\big[(1\otimes 1\otimes S)\phi_F^{-1}\big]_{124}\beta_{F3}\\
&= \big[(1\otimes 1\otimes S)(F\otimes 1)\cdot(\Delta\otimes \mathrm{id}_H)(F)\cdot\phi^{-1}\cdot(\mathrm{id}_H\otimes\Delta)(F^{-1})\cdot(1\otimes F^{-1})\big]_{124}\beta_{F3}\\
&= \big[(1\otimes 1\otimes S)(F\otimes 1)\cdot(\Delta\otimes \mathrm{id}_H)(F)\cdot\phi^{-1}\cdot(1\otimes F^{-1})\cdot(\mathrm{id}_H\otimes\Delta_F)(F^{-1})\big]_{124}\beta_{F3}\\
&= \big[(1\otimes 1\otimes S)(F\otimes 1)\cdot(\Delta\otimes \mathrm{id}_H)(F)\cdot\phi^{-1}\cdot(1\otimes F^{-1})\big]_{124}\beta_{F3}\\
&= \big[(1\otimes 1\otimes S)(F\otimes 1)\cdot(\Delta\otimes \mathrm{id}_H)(F)\cdot\phi^{-1}\big]_{124}\beta_{3}\ .
\end{aligned}
\tag{2.2.102}
$$

The first equality follows from (2.1.109), the second equality follows from (2.1.108), the third equality follows from (2.1.100b) and (2.1.106), and the fourth equality follows from (2.1.110). Now including the representations in (2.2.38) we obtain the result (cf. (2.2.47) and (2.2.25)). □

Since for H a quasi-Hopf algebra and $F \in H\otimes H$ a cochain twist based on H, H_F is a quasi-Hopf algebra (cf. Theorem 2.1.40), there exist $[H_F, \mathscr{M}]$-morphisms $\mathrm{ev}^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W)}$ and $\bullet^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W),\mathcal{F}(\rho_X)}$ by Proposition 2.1.5, for any three objects $\mathcal{F}(\rho_V), \mathcal{F}(\rho_W), \mathcal{F}($ in the closed monoidal category $[H_F, \mathscr{M}]$. These morphisms are related to the corresponding $[H, \mathscr{M}]$-morphisms $\mathrm{ev}_{\rho_V,\rho_W}$ and $\bullet_{\rho_V,\rho_W,\rho_X}$ by

Proposition 2.2.35. *If ρ_V, ρ_W, ρ_X are any three objects in $[H, \mathscr{M}]$, then the diagrams*

$$
\begin{tikzcd}
\rho_{\mathrm{hom}_F(\mathcal{F}(V),\mathcal{F}(W))}\otimes_F \mathcal{F}(\rho_V) \arrow[rr, Rightarrow, "\mathrm{ev}^F"] \arrow[d, Rightarrow, "\gamma\otimes_F \mathrm{id}"'] & & \mathcal{F}(\rho_W)\\
\mathcal{F}\big(\rho_{\mathrm{hom}(V,W)}\big)\otimes_F \mathcal{F}(\rho_V) \arrow[d, Rightarrow, "\varphi"'] & & \\
\mathcal{F}\big(\rho_{\mathrm{hom}(V,W)}\otimes\rho_V\big) \arrow[uurr, Rightarrow, "\mathcal{F}(\mathrm{ev})"'] & &
\end{tikzcd}
\tag{2.2.103a}
$$

$$\begin{tikzcd}
\rho_{\hom_F(\mathcal{F}(W),\mathcal{F}(X))} \otimes_F \rho_{\hom_F(\mathcal{F}(V),\mathcal{F}(W))} \arrow[rr, Rightarrow, "\bullet^F"] \arrow[d, Rightarrow, "\gamma\otimes_F\gamma"'] & & \rho_{\hom_F(\mathcal{F}(V),\mathcal{F}(X))} \arrow[dd, Rightarrow, "\gamma"] \\
\mathcal{F}\big(\rho_{\hom(W,X)}\big) \otimes_F \mathcal{F}\big(\rho_{\hom(V,W)}\big) \arrow[d, Rightarrow, "\varphi"'] & & \\
\mathcal{F}\big(\rho_{\hom(W,X)} \otimes \rho_{\hom(V,W)}\big) \arrow[rr, Rightarrow, "\mathcal{F}(\bullet)"'] & & \mathcal{F}\big(\rho_{\hom(V,X)}\big)
\end{tikzcd} \tag{2.2.103b}$$

in $[H, \mathscr{M}]$ commute. That is (without subscripts)

$$\mathrm{ev}^F = \mathcal{F}(\mathrm{ev}) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \ , \tag{2.2.104a}$$

$$\bullet^F = \gamma^{-1} \circ \mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma) \ . \tag{2.2.104b}$$

Proof. (2.2.104a) is derived in a similar way to (2.2.101): Starting with $\mathrm{ev}^F = {\zeta_F}^{-1}(\mathrm{id})$ (cf. (2.2.41)) in $[H_F, \mathscr{M}]$ we notice that the properties of the quasi-antipode, coproduct and associator in H_F are such that:

$$\begin{aligned}
&[(1 \otimes S \otimes 1)(\phi_F)]_{124}\, \alpha_{F3} \\
&= [(1 \otimes S \otimes 1)((1 \otimes F) \cdot (1 \otimes \Delta)(F) \cdot \phi \cdot (\Delta \otimes 1)(F^{-1}) \cdot (F^{-1} \otimes 1))]_{124}\, \alpha_{F3} \\
&= [(1 \otimes S \otimes 1)((1 \otimes \Delta_F)(F) \cdot (1 \otimes F) \cdot \phi \cdot (\Delta \otimes 1)(F^{-1}) \cdot (F^{-1} \otimes 1))]_{124}\, \alpha_{F3} \\
&= [(1 \otimes S \otimes 1)(\phi \cdot (\Delta \otimes 1)(F^{-1}) \cdot (F^{-1} \otimes 1))]_{124}\, \alpha_3 \ .
\end{aligned} \tag{2.2.105}$$

The second equality follows from (2.1.108), and the third equality follows from (2.1.100a) in $[H_F, \mathscr{M}]$, i.e. $S(h_{(1)_F})\, \alpha_F\, h_{(2)_F} = \epsilon(h)\, \alpha_F$ (with h equal to the second leg of the cochain twist), the counitality of the twist (2.1.106), and the definition of α_F in terms of α (2.1.110). Now including the representations in (2.2.41) we obtain the result (cf. (2.2.47) and (2.2.25)).

In order to prove commutativity of the second diagram, first notice that, due to Proposition 2.2.13 (i) and the bijectivity of the currying maps, it is enough to show that

$${\zeta_F}^{-1}(\bullet^F) = {\zeta_F}^{-1}\big(\gamma^{-1} \circ \mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma)\big) \ . \tag{2.2.106}$$

After using Proposition 2.2.13 (i) in the category $[H_F, \mathscr{M}]$ together with equation (2.2.104a) and cancelling an instance of γ with its inverse, the right hand side of (2.2.106) is equal to

$$
\begin{aligned}
&= \mathcal{F}(\mathrm{ev}) \circ \varphi \circ \big(\mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma) \otimes_F \mathrm{id}\big) \\
&= \mathcal{F}(\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \Phi) \circ \varphi \circ \big(\varphi \circ (\gamma \otimes_F \gamma) \otimes_F \mathrm{id}\big) \\
&= \mathcal{F}(\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev})) \circ \varphi \circ (\mathrm{id} \otimes \varphi) \circ \Phi^F \circ \big(\gamma \otimes_F (\gamma \otimes_F \mathrm{id})\big) \\
&= \mathcal{F}(\mathrm{ev}) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \circ \big(\mathrm{id} \otimes_F \mathcal{F}(\mathrm{ev}) \circ \varphi \circ (\gamma \otimes_F \mathrm{id})\big) \circ \Phi^F \\
&= \mathrm{ev}^F \circ (\mathrm{id} \otimes \mathrm{ev}^F) \circ \Phi^F \ .
\end{aligned} \tag{2.2.107}
$$

The second equality follows from the H_F-equivariance of $\mathcal{F}(\bullet)$ and the property 2.2.58 of the internal composition. The third equality follows from the definition of the twisted associator (2.1.109). The fourth equality follows from the H_F-equivariance of γ and ev and the last equality follows by using (2.2.104a). The final equality is equal to ${\zeta_F}^{-1}(\bullet^F)$ which proves (2.2.104b). □

The cochain twisting of the internal commutator is derived from that of the braiding and the internal composition.

Lemma 2.2.36. *The braiding natural transformations and internal commutators in the closed braided monoidal categories $[H, \mathscr{M}]$ and $[H_F, \mathscr{M}]$ are related by*

$$\tau_F = \varphi^{-1} \circ \mathcal{F}(\tau) \circ \varphi \ , \tag{2.2.108a}$$

$$[\,\cdot\,,\,\cdot\,]_F = \gamma^{-1} \circ \mathcal{F}\big([\,\cdot\,,\,\cdot\,]\big) \circ \varphi \circ (\gamma \otimes_F \gamma) \ . \tag{2.2.108b}$$

Proof. Equation (2.2.108a) follows directly from the definition of the twisted quasi-triangular R-matrix $R_F = F_{21} R F^{-1}$ (cf. (2.1.112)). The equality (2.2.108b) follows from (2.2.104b) and (2.2.108a)

$$
\begin{aligned}
[\,\cdot\,,\,\cdot\,]_F &= \bullet_F - \bullet_F \circ \tau_F \\
&= \gamma^{-1} \circ \mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma)(\mathrm{id} \otimes_F \mathrm{id} - \varphi^{-1} \circ \mathcal{F}(\tau) \circ \varphi) \\
&= \gamma^{-1} \circ \mathcal{F}\big([\,\cdot\,,\,\cdot\,]\big) \circ \varphi \circ (\gamma \otimes_F \gamma) \ , \qquad (2.2.109)
\end{aligned}
$$

where the final step follows from the H_F-equivariance of $\gamma \otimes_F \gamma$. □

For the braided closed monoidal category $[H_F, \mathscr{M}]$ one has by Proposition 2.1.7 the internal tensor product $\circledast^F$ for the internal hom-objects hom_F. It is related to the corresponding internal tensor product $\circledast$ in $[H, \mathscr{M}]$ by

Proposition 2.2.37. *If $\rho_V, \rho_W, \rho_X, \rho_Y$ are any four objects in $[H, \mathscr{M}]$, then the diagram in (2.2.111) commutes.*

Proof. The strategy for this proof is similar to that of the proof of Proposition 2.2.30. In the special case where the objects X and Y are the same, one can prove directly that the diagram in (2.2.111) commutes when acting on elements of the form $L \otimes_F 1_F$; this computation makes use of Proposition 2.2.35 to express ev^F and $\mathcal{F}(\mathrm{ev})$ in terms of each other. Similarly, one can prove that in the case where the objects V and W are the same the diagram in (2.2.111) commutes when acting on elements of the form $1_F \otimes_F L'$. In the generic situation recall that by Lemma 2.2.28 one has

$$
L \circledast^F L' = \big(L \circledast^F 1_F\big) \bullet^F \big(1_F \circledast^F L'\big) \ , \qquad (2.2.110)
$$

which reduces the problem of proving commutativity of the diagram in (2.2.111) to the two special cases above. The relevant step here is to use Proposition 2.2.35 in order to express $\bullet^F$ and $\mathcal{F}(\bullet)$ in terms of each other. □

$$
\begin{tikzcd}
\hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \otimes_F \hom_F(\mathcal{F}(\rho_X), \mathcal{F}(\rho_Y)) \arrow[rr, Rightarrow, "\odot^F_{\mathcal{F}(\rho_V), \mathcal{F}(\rho_W), \mathcal{F}(\rho_X), \mathcal{F}(\rho_Y)}"] \arrow[d, Rightarrow, "\gamma_{\rho_V, \rho_W} \otimes_F \gamma_{\rho_X, \rho_Y}"'] & & \hom_F(\mathcal{F}(\rho_V) \otimes_F \mathcal{F}(\rho_X), \mathcal{F}(\rho_W) \otimes_F \mathcal{F}(\rho_Y)) \arrow[d, Rightarrow, "\varphi_{\rho_W, \rho_Y} \circ (\,\cdot\,) \circ \varphi^{-1}_{\rho_V, \rho_X}"] \\
\mathcal{F}(\hom(\rho_V, \rho_W)) \otimes_F \mathcal{F}(\hom(\rho_X, \rho_Y)) \arrow[d, Rightarrow, "\varphi_{\hom(\rho_V, \rho_W), \hom(\rho_X, \rho_Y)}"'] & & \hom_F(\mathcal{F}(\rho_V \otimes \rho_X), \mathcal{F}(\rho_W \otimes \rho_Y)) \arrow[d, Rightarrow, "\gamma_{\rho_V \otimes \rho_X, \rho_W \otimes \rho_Y}"] \\
\mathcal{F}(\hom(\rho_V, \rho_W) \otimes \hom(\rho_X, \rho_Y)) \arrow[rr, Rightarrow, "\mathcal{F}(\odot_{\rho_V, \rho_W, \rho_X, \rho_Y})"'] & & \mathcal{F}(\hom(\rho_V \otimes \rho_X, \rho_W \otimes \rho_Y))
\end{tikzcd}
\tag{2.2.111}
$$

2.2.15 H-invariant internal homomorphisms

In this thesis the notion of internal homomorphism is central. This is a consequence of the result which we shall show in this subsection that the internal homomorphisms in $[H,\mathscr{M}]$ extend the morphism sets in $[H,\mathscr{M}]$ in a structure preserving way.

Given any object (ρ_V,ρ_W) in $\big([H,\mathscr{M}]\big)^{\mathrm{op}}\times[H,\mathscr{M}]$, one can assign to it the set of H-invariant internal homomorphisms

$$\hom^H(\rho_V,\rho_W)(*) := \big\{L\in \mathrm{Hom}_{\mathscr{M}}(V,W) : \rho_{\hom(V,W)}(h)(L)=\epsilon(h)\,L\,,\ \forall h\in H\big\}\ . \tag{2.2.112}$$

$\hom^H : \big([H,\mathscr{M}]\big)^{\mathrm{op}}\times[H,\mathscr{M}]\to \mathsf{Sets}$ is a functor (in fact, it is a subfunctor of the internal hom-functor composed with the forgetful functor from $[H,\mathscr{M}]$ to the category of sets): we have by the H-equivariance of g

$$\rho_{\hom(V,W)}(h)(g\circ L\circ f)=\epsilon(h)\,g\circ L\circ f\ . \tag{2.2.113}$$

Furthermore we notice that the functor $\hom^H$ has the same source and target as the functor $\mathrm{Hom} : \big([H,\mathscr{M}]\big)^{\mathrm{op}}\times[H,\mathscr{M}]\to\mathsf{Sets}$ assigning the morphism sets. The next proposition shows that the morphisms in $[H,\mathscr{M}]$ can be identified with the H-invariant internal homomorphisms.

Proposition 2.2.38. *Let H be a quasitriangular quasi-Hopf algebra.*

(i) There is a natural isomorphism $\vartheta : \mathrm{Hom}_{[H,\mathscr{M}]}\Rightarrow\hom^H$ of functors from $\big([H,\mathscr{M}]\big)^{\mathrm{op}}\times[H,\mathscr{M}]$ to Sets. Explicitly, the (ρ_V,ρ_W)-component of ϑ is given by

$$\vartheta_{\rho_V,\rho_W} : \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V,\rho_W)\Longrightarrow\hom^H(\rho_V,\rho_W)\ ,\quad f\longmapsto\rho_W(\beta)\circ f\ , \tag{2.2.114}$$

with single component

$$\vartheta_{V,W} : \mathrm{Hom}_{\mathscr{M}}(V,W)\longrightarrow\mathrm{Hom}_{\mathscr{M}}(V,W)\ ,\quad f\longmapsto\rho_W(\beta)\circ f\ , \tag{2.2.115}$$

with the property that

$$\mathrm{ev}_{V,W}\big(\vartheta_{V,W}(f) \otimes v\big) = f(v) \ , \tag{2.2.116}$$

for all $f \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V, \rho_W)$ *and* $v \in V$.

(ii) The natural isomorphism $\vartheta : \mathrm{Hom}_{[H,\mathscr{M}]} \Rightarrow \mathrm{hom}^H$ *preserves compositions and tensor products, i.e. there are identities*

$$\bullet_{V,W,X} \circ (\vartheta_{W,X} \otimes \vartheta_{V,W}) = \vartheta_{V,X} \circ (\circ) \ , \tag{2.2.117a}$$

$$\circledast_{V,W,X,Y} \circ (\vartheta_{V,W} \otimes \vartheta_{X,Y}) = \vartheta_{V\otimes X, W\otimes Y} \circ (\otimes) \ . \tag{2.2.117b}$$

(iii) For all $f \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V, \rho_W)$*,* $g \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_W, \rho_X)$*,* $L' \in \mathrm{hom}(V,W)$ *and* $L \in \mathrm{hom}(W,X)$ *one has*

$$\vartheta_{W,X}(g) \bullet_{V,W,X} L' = g \circ L' \quad , \qquad L \bullet_{V,W,X} \vartheta_{V,W}(f) = L \circ f \ . \tag{2.2.118}$$

Proof. It is easy to see that $\vartheta_{V,W}(f)$ is H-invariant for any $f \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V, \rho_W)$: for all $h \in H$ one has by a similar calculation to (2.2.74)

$$\begin{aligned}
\rho_{\mathrm{hom}(V,W)}(h)\big(\vartheta_{V,W}(f)\big) &= \rho_W(h_{(1)}) \circ \rho_W(\beta) \circ f \circ \rho_V(S(h_{(2)})) \\
&= \rho_W(h_{(1)}\, \beta\, S(h_{(2)})) \circ f \\
&= \epsilon(h)\rho_V(\beta) \circ f \\
&= \epsilon(h)\, \vartheta_{V,W}(f) \ ,
\end{aligned} \tag{2.2.119}$$

where the second equality follows from the naturality of f and the functoriality of representations and the third equality follows by property (2.1.100b). One can now show that the map $\vartheta_{V,W}$ is invertible via

$$\vartheta^{-1}_{\rho_V,\rho_W} : \mathrm{hom}^H(\rho_V, \rho_W) \Longrightarrow \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V, \rho_W) \tag{2.2.120}$$

with single component

$$\begin{aligned}
\vartheta^{-1}_{V,W} : \mathrm{Hom}_{\mathscr{M}}(V,W) &\longrightarrow \mathrm{Hom}_{\mathscr{M}}(V,W) \ , \\
L &\longmapsto \mathrm{ev}(L\otimes -) \ .
\end{aligned} \tag{2.2.121}$$

Since ev is H-equivariant it is easy to see that $\vartheta^{-1}_{V,W}(L)$ is H-equivariant for any H-invariant $L \in \mathrm{Hom}_{\mathscr{M}}(V,W)$: for all $h\in H$ one has

$$\rho_W(h)\big(\mathrm{ev}(L\otimes -)\big) = \mathrm{ev}(\rho_{\mathrm{hom}(V,W)}(h_{(1)})(L)\otimes \rho_V(h_{(2)})) = \mathrm{ev}(L\otimes -)\circ \rho_V(h) \ , \tag{2.2.122}$$

since $\epsilon(h_{(1)})\, h_{(2)} = h$. The fact that $\vartheta^{-1}_{V,W}$ is the inverse of $\vartheta_{V,W}$ can be checked as follows: by a similar calculation to (2.2.73b) one has

$$\begin{aligned}
\vartheta^{-1}_{V,W}\circ \vartheta_{V,W}(f) &= \mathrm{ev}(\vartheta_{V,W}(f)\otimes -) \\
&= \rho_W(\phi^{(1)})\circ \rho_W(\beta)\circ f\circ \rho_V\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\big) \\
&= f \ ,
\end{aligned} \tag{2.2.123}$$

for all $f\in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V,\rho_W)$, where the final step follows from the naturality of f, the functoriality of representations and the property (2.1.100c). For any H-invariant $L\in \mathrm{Hom}_{\mathscr{M}}(V,W)$ one has

$$\begin{aligned}
\vartheta_{V,W}\circ \vartheta^{-1}_{V,W}(L) &= \rho_W(\beta)\circ \mathrm{ev}(L\otimes -) \\
&= \mathrm{ev}(L\otimes \rho_V(\beta)) \\
&= \rho_W(\phi^{(1)})\circ L\circ \rho_V(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\,\beta) \\
&= \rho_W(\phi^{(1)})\circ \rho_{\mathrm{hom}(V,W)}(\widetilde{\phi}^{(-1)})\circ L\circ \rho_V(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\,\widetilde{\phi}^{(-2)}\,\beta\, S(\widetilde{\phi}^{(-3)})) \\
&= \rho_W(\phi^{(1)}\,\widetilde{\phi}^{(-1)}_{(1)})\circ L\circ \rho_V(S(\phi^{(2)}\,\widetilde{\phi}^{(-1)}_{(2)})\,\alpha\,\phi^{(3)}\,\widetilde{\phi}^{(-2)}\,\beta\, S(\widetilde{\phi}^{(-3)})) \\
&= L\circ \rho_V(S(\phi^{(-1)})\,\alpha\,\phi^{(-2)}\,\beta\, S(\phi^{(-3)})) \\
&= L \ .
\end{aligned} \tag{2.2.124}$$

The second equality follows from (2.2.122), the third equality follows from (2.2.112) and $\epsilon(\widetilde{\phi}^{(-1)})\,\widetilde{\phi}^{(-2)} \otimes \widetilde{\phi}^{(-3)} = 1 \otimes 1$, and the fourth equality follows from applying (2.1.98c) and then using (2.1.100a), (2.1.100b) and (2.1.99) to eliminate two of the three factors of ϕ.

For item (ii) one has

$$\begin{aligned}
\bullet_{V,W,X} \circ (\vartheta_{W,X} \otimes \vartheta_{V,W}) &= \zeta(\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \Phi) \circ (\vartheta_{W,X} \otimes \vartheta_{V,W}) \\
&= \mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ (\vartheta_{W,X} \otimes \vartheta_{V,W} \otimes \rho_V(\beta)) \\
&= (\circ) \circ \rho_V(\beta) \\
&= \vartheta_{V,X} \circ (\circ) \ .
\end{aligned} \tag{2.2.125}$$

Here the second equality follows from the H-invariance of the image of ϑ, the third equality follows from result (2.2.116) and the fourth by the naturality of $[H, \mathscr{M}]$-morphisms. By a similar calculation one has $\circledbullet_{V,W,X,Y} \circ (\vartheta_{V,W} \otimes \vartheta_{X,Y}) = \vartheta_{V\otimes X, W\otimes Y} \circ (\otimes)$.

For (iii) one has

$$\begin{aligned}
\vartheta(g) \bullet L' &= \zeta(\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \Phi) \circ (\vartheta(g) \otimes L') \\
&= \mathrm{ev}(\vartheta(g) \otimes \mathrm{ev} \circ \zeta(\mathrm{id})(L')) \\
&= g \circ \mathrm{ev} \circ (\zeta(\mathrm{id})(L') \otimes \mathrm{id}) \\
&= g \circ \zeta^{-1}(\zeta(\mathrm{id}))(L') \\
&= g \circ L' \ ,
\end{aligned} \tag{2.2.126}$$

for all $g \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_W, \rho_X)$ and $L' \in \mathrm{hom}(V, W)$, where the second equality follows from the H-invariance of $\vartheta(g)$ and property (2.1.98a), the third equality follows from the property (2.2.116) and the fourth equality follows from Proposition 2.2.13 (i). By a completely analogous calculation one has $L \bullet \vartheta(f) = L \circ f$ for all $f \in \mathrm{Hom}_{[H,\mathscr{M}]}(\rho_V, \rho_W)$ and $L \in \mathrm{hom}(W, X)$. □

Remark 2.2.39. Both functors $\mathrm{Hom}_{\mathscr{M}}$ and hom^H can be promoted to functors with values in the category of k-modules $\mathscr{M}$. The components of the natural isomorphism ϑ in Proposition 2.2.38 are obviously k-linear isomorphisms, hence ϑ also gives a natural isomorphism between $\mathrm{Hom}_{\mathscr{M}}$ and $\hom^H$ when considered as functors with values in $\mathscr{M}$.

2.3 The subcategory of symmetric bimodules

The sections of a vector bundle over a classical manifold form a symmetric bimodule over the algebra of functions on the manifold. We shall see in this section that twist deformation quantisation preserves the symmetry of the left and right bimodule actions. Since our aim is to provide descriptions of differential geometry for spaces of the type arising from twist deformation quantisation of classical manifolds we focus attention on the subcategory of symmetric bimodules over an algebra object in $[H, \mathscr{M}]$.

Fixing any quasitriangular quasi-Hopf algebra H and any algebra ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$, the category H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ of symmetric ρ_A-bimodules in $[H, \mathscr{M}]$ is a closed braided monoidal category. In this section we show how all structures in the closed braided monoidal category $[H, \mathscr{M}]$ systematically descend to the closed braided monoidal category H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ of symmetric bimodule objects over some algebra ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$. Cochain twisting leads to an equivalence between the closed braided monoidal categories H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ and H_F–$\mathrm{Bimod}(A_F)^{\mathrm{sym}}$, where A_F is a commutative algebra in $[H_F, \mathscr{M}]$ which is given by a cochain twist of the original algebra A.

2.3.1 The category of symmetric bimodules

In the following let us fix a quasitriangular quasi-Hopf algebra H and denote the R-matrix by $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$.

Definition 2.3.1 (Symmetric bimodule)**.** Let H be a quasitriangular quasi-Hopf algebra and let ρ_A be an algebra in H-$\mathrm{Alg}^{\mathrm{com}}$. A *symmetric* A-bimodule in $[H, \mathscr{M}]$ is an object ρ_V in $[H, \mathscr{M}]$ together with two $[H, \mathscr{M}]$-morphisms $l_V : \rho_A \otimes \rho_V \Rightarrow \rho_V$

(left ρ_A-action) and $r_V = l_V \circ \tau_{V,A}$ (right ρ_A-action), such that the following diagrams

$$\begin{tikzcd}
\rho_A \otimes (\rho_A \otimes \rho_V) \arrow[r, Rightarrow, "\mathrm{id}_{\rho_A} \otimes l_{\rho_V}"] \arrow[d, Rightarrow, "\Phi^{-1}_{\rho_A,\rho_A,\rho_V}"'] & \rho_A \otimes \rho_V \arrow[dd, Rightarrow, "l_{\rho_V}"] \\
(\rho_A \otimes \rho_A) \otimes \rho_V \arrow[d, Rightarrow, "\mu_{\rho_A} \otimes \mathrm{id}_{\rho_V}"'] & \\
\rho_A \otimes \rho_V \arrow[r, Rightarrow, "l_{\rho_V}"'] & \rho_V
\end{tikzcd} \tag{2.3.1a}$$

$$\begin{tikzcd}
\rho_I \otimes \rho_V \arrow[d, Rightarrow, "\eta_{\rho_A} \otimes \mathrm{id}_{\rho_V}"'] \arrow[dr, Rightarrow, "\lambda_{\rho_V}"] & \\
\rho_A \otimes \rho_V \arrow[r, Rightarrow, "l_{\rho_V}"'] & \rho_V
\end{tikzcd} \tag{2.3.1b}$$

commute.

Definition 2.3.2. [Category of symmetric bimodules] Let H be a quasitriangular quasi-Hopf algebra and let ρ_A be an algebra in $H\text{-}\mathrm{Alg}^{\mathrm{com}}$. The collection of symmetric ρ_A-bimodules in $[H, \mathscr{M}]$ together with $[H, \mathscr{M}]$-morphisms $f : \rho_V \Rightarrow \rho_W$ which preserve the left (and hence automatically also the right) ρ_A-actions l_V and r_V, i.e. such that

$$f \circ l_V = l_W \circ (\mathrm{id}_{\rho_A} \otimes_I f) \ , \tag{2.3.2}$$

constitute a subcategory of $[H, \mathscr{M}]$. This subcategory is equal to the comma category $(\otimes_{\rho_I} \Rightarrow \mathrm{id}_{[H,\mathscr{M}]})$ whose objects are triples $(\rho_A \times \rho_V, l_V, \rho_V)$ with $(\rho_V, l_V, l_V \circ \tau_{V,A})$ a symmetric A-bimodule in $[H, \mathscr{M}]$ and whose morphisms are tuples of morphisms $(\mathrm{id}_{\rho_A} \otimes_{\rho_I} f, f)$ satisfying (2.3.2). We shall denote by

$$H\text{-}\mathrm{Bimod}(A)^{\mathrm{sym}} \ , \tag{2.3.3}$$

the *category of symmetric ρ_A-bimodules* in $[H, \mathscr{M}]$. And with an abuse of notation denote objects in $H\text{-}\mathrm{Bimod}(A)^{\mathrm{sym}}$ by the corresponding objects in $[H, \mathscr{M}]$.

Remark 2.3.3. If $f : \rho_V \Rightarrow \rho_W$ is an H-Bimod$(A)^{\mathrm{sym}}$-morphism between two objects ρ_V, ρ_W in H-Bimod$(A)^{\mathrm{sym}}$, then

$$\begin{aligned} f \circ r_V &= f \circ l_V \circ \tau_{V,A} \\ &= l_W \circ (\mathrm{id}_A \otimes f) \circ \tau_{V,A} \\ &= l_W \circ \tau_{W,A} \circ (f \otimes \mathrm{id}_A) \\ &= r_W \circ (f \otimes \mathrm{id}_A) \ . \end{aligned} \tag{2.3.4}$$

The second equality follows from the left ρ_A-linearity of f and the third equality follows from the H-equivariance of f. This shows that any $[H, \mathscr{M}]$-morphism of objects in H-Bimod$(A)^{\mathrm{sym}}$ preserving the left ρ_A-action automatically also preserves the right ρ_A-action.

The right ρ_A-action in a symmetric ρ_A-bimodule ρ_V in $[H, \mathscr{M}]$ is defined in terms of the left ρ_A-action by

$$r_V = l_V \circ \tau_{V,A} = l_V \circ (\rho_A \otimes \rho_V)(R_{21}) \circ \sigma \ . \tag{2.3.5}$$

The first diagram in Definition 2.1.16 implies that

$$l_V \circ (\mathrm{id}_A \otimes l_V) = l_V \circ (\mu_A \otimes \mathrm{id}_V) \circ \Phi^{-1}_{A,A,V} \ , \tag{2.3.6a}$$

$$r_V \circ (r_V \otimes \mathrm{id}_A) = l_V \circ (\mathrm{id}_V \otimes \mu_A) \circ \Phi_{V,A,A} \ , \tag{2.3.6b}$$

$$l_V \circ (\mathrm{id}_A \otimes r_V) = r_V \circ (l_V \otimes \mathrm{id}_A) \circ \Phi^{-1}_{A,V,A} \ . \tag{2.3.6c}$$

The last two equations follow from definition of the right ρ_A-action in terms of the left ρ_A-action and properties of the R-matrix. These are weak versions (i.e. up to associator) of the usual bimodule properties (cf. Definition 2.1.16).

Example 2.3.4. The n-dimensional free A-bimodule A^n in $\mathscr{M}$ of Example 2.1.17 becomes a symmetric ρ_A-bimodule ρ_A^n in $[H, \mathscr{M}]$ when A is a braided commutative algebra in $[H, \mathscr{M}]$ with left H-action defined componentwise by

$$\rho_{A^n}(h)\vec{a} := \begin{pmatrix} \rho_A(h)a_1 \\ \vdots \\ \rho_A(h)a_n \end{pmatrix} , \tag{2.3.7}$$

for all $h \in H$ and $\vec{a} \in \rho_A^n(*) = A^n$.

2.3.2 Cochain twisting the category

In complete analogy to the result in Subsection 2.2.11 we find that the categories H-Bimod$(A)^{\mathrm{sym}}$ and H_F-Bimod$(A_F)^{\mathrm{sym}}$ are equivalent for any quasitriangular quasi-Hopf algebra H, braided commutative algebra ρ_A in $[H, \mathscr{M}]$ and cochain twist $F \in H \otimes H$. Using the monoidal functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ we obtain for any object ρ_V in H-Bimod$(A)^{\mathrm{sym}}$ an object $\mathcal{F}(\rho_V)$ in $[H_F, \mathscr{M}]$. For this object we define the $[H_F, \mathscr{M}]$-morphism $l_{V_F} : \mathcal{F}(\rho_A) \otimes_F \mathcal{F}(\rho_V) \Rightarrow \mathcal{F}(\rho_V)$ via the coherence maps (2.2.24) and the diagram

$$\begin{tikzcd} \mathcal{F}(\rho_A) \otimes_F \mathcal{F}(\rho_V) \arrow[r, Rightarrow, "l_{\rho_{V_F}}"] \arrow[d, Rightarrow, "\varphi_{\rho_A,\rho_V}"'] & \mathcal{F}(\rho_V) \\ \mathcal{F}(\rho_A \otimes \rho_V) \arrow[ur, Rightarrow, "\mathcal{F}(l_{\rho_V})"'] & \end{tikzcd} \tag{2.3.8}$$

in $[H_F, \mathscr{M}]$ and the $[H_F, \mathscr{M}]$-morphism r_{V_F} by

$$r_{V_F} := l_{V_F} \circ \tau^F_{\mathcal{F}(V),\mathcal{F}(A)} . \tag{2.3.9}$$

It is straightforward to check that $\mathcal{F}(\rho_V)$, together with the $[H_F, \mathscr{M}]$-morphisms l_{V_F} and r_{V_F} is an $\mathcal{F}(\rho_A)$-bimodule in $[H_F, \mathscr{M}]$: Using

$$l_{V_F} = l_V \circ \varphi_{A,V} , \tag{2.3.10}$$

$$r_{V_F} = r_V \circ \varphi_{V,A} \tag{2.3.11}$$

((2.3.11) follows from a short calculation using $R_F = F_{21}\, R\, F^{-1}$) we have by (2.3.8) and the definition of the associator in H_F in terms of the associator in H that

$$l_{V_F} \circ (\mathrm{id}_{A_F} \otimes l_{V_F}) = l_{V_F} \circ (\mu_{A_F} \otimes \mathrm{id}_{V_F}) \circ ((\rho_A \otimes \rho_A) \otimes \rho_V)(\phi_F^{-1}) \ , \tag{2.3.12}$$

and, since R_F is an R-matrix for the quasitriangular quasi-Hopf algebra H_F, that the last two conditions in (2.3.6) are satisfied with the right A_F-action (2.3.9). By the counital condition of the twist we have that

$$l_{V_F} \circ (\eta_{A_F} \otimes \mathrm{id}_{V_F}) = \lambda_{V_F} \ . \tag{2.3.13}$$

We shall denote this A_F-bimodule also by $\mathcal{F}(\rho_V)(*) = V_F$. If $f : \rho_V \Rightarrow \rho_W$ is an H-Bimod$(A)^{\mathrm{sym}}$-morphism, then the $[H_F, \mathscr{M}]$-morphism $\mathcal{F}(f) : \mathcal{F}(\rho_V) \Rightarrow \mathcal{F}(\rho_W)$ preserves the left (and right) $\mathcal{F}(\rho_A)$-action, i.e. it is an H_F-Bimod$(A_F)^{\mathrm{sym}}$-morphism with single component (denoted by the same symbol) $\mathcal{F}(f) : V_F \to W_F$. In summary, we have shown

Proposition 2.3.5. *If H is a quasitriangular quasi-Hopf algebra, ρ_A is a braided commutative algebra in $[H, \mathscr{M}]$ and $F \in H \otimes H$ is any cochain twist based on H, then the categories H-*Bimod$(A)^{\mathrm{sym}}$ *and H_F-*Bimod$(A_F)^{\mathrm{sym}}$ *are equivalent, where $\mathcal{F}(\rho_A)(*) = A_F$ is the algebra obtained by applying the functor described in Proposition 2.2.24 on ρ_A.*

2.3.3 The monoidal structure

Interpreting the theory of Subsection 2.1.4 in the monoidal category $[H, \mathscr{M}]$ makes sense:

Taking ρ_A to be a commutative algebra in H–Alg$^{\mathrm{com}}$, the properties of a quasitriangular quasi-Hopf algebra are such that objects $\rho_V \otimes \rho_W \in [H, \mathscr{M}]$, where $(\rho_V, \rho_W) \in H\text{-Bimod}(A)^{\mathrm{sym}} \times H\text{-Bimod}(A)^{\mathrm{sym}}$, obtained by using the functor

$$\otimes \circ (\mathsf{Forget} \times \mathsf{Forget}) : H\text{-Bimod}(A)^{\mathrm{sym}} \times H\text{-Bimod}(A)^{\mathrm{sym}} \longrightarrow [H, \mathscr{M}] \ . \tag{2.3.14}$$

with the forgetful functor $\mathsf{Forget} : H\text{-Bimod}(A)^{\mathrm{sym}} \to [H, \mathscr{M}]$, can be equipped with the structure of a symmetric bimodule object in $[H, \mathscr{M}]$ with left and right ρ_A-actions given by the $[H, \mathscr{M}]$-morphisms

$$l_{V\otimes W} = (l_V \otimes \mathrm{id}_W) \circ ((\rho_A \otimes \rho_V) \otimes \rho_W)(\phi^{-1}) \ , \tag{2.3.15a}$$

$$\begin{aligned} r_{V\otimes W} &:= l_{V\otimes W} \circ \tau_{V\otimes W, A} \\ &= l_{V\otimes W} \circ (\rho_A \otimes (\rho_V \otimes \rho_W))(\mathrm{id}_H \otimes \Delta)(R_{21}) \circ \sigma \ . \end{aligned} \tag{2.3.15b}$$

Using (2.1.98c) one easily checks that $l_{V\otimes W}$ satisfies the properties of a left A-action. The right and left-right properties in Definition 2.1.16 follow by the same calculations as for (2.3.6). Also given a morphism $\big(f : \rho_V \Rightarrow \rho_X, g : \rho_W \Rightarrow \rho_Y\big)$ in $H\text{-Bimod}(A)^{\mathrm{sym}} \times H\text{-Bimod}(A)^{\mathrm{sym}}$, it is clear that the $[H, \mathscr{M}]$-morphism $f \otimes g : \rho_V \otimes \rho_W \Rightarrow \rho_X \otimes \rho_Y$ preserves this ρ_A-bimodule structure, i.e. it is a morphism in $H\text{-Bimod}(A)^{\mathrm{sym}}$. So the functor in (2.3.14) is promoted to a functor with values in $H\text{-Bimod}(A)^{\mathrm{sym}}$.

Furthermore the properties of a quasitriangular quasi-Hopf algebra are such that the properties in Lemma 2.1.20 hold when interpreted in the category $[H, \mathscr{M}]$: Item (i) follows by using (2.1.98c) and item (ii) follows by the bimodule properties of an $H\text{-Bimod}(A)^{\mathrm{sym}}$-object. So one obtains the monoidal functor for the category $H\text{-Bimod}(A)^{\mathrm{sym}}$

$$\otimes_A : H\text{-Bimod}(A)^{\mathrm{sym}} \times H\text{-Bimod}(A)^{\mathrm{sym}} \to H\text{-Bimod}(A)^{\mathrm{sym}} \tag{2.3.16}$$

which assigns to any two objects ρ_V, ρ_W in $H\text{-Bimod}(A)^{\mathrm{sym}}$ the object

$$\rho_V \otimes_A \rho_W = \frac{\rho_V \otimes \rho_W}{\mathrm{Im}\big(r_{\rho_V} \otimes \mathrm{id} - (\mathrm{id} \otimes l_{\rho_W}) \circ \Phi_{\rho_V, \rho_A, \rho_W}\big)} \tag{2.3.17}$$

in $[{}_H\mathscr{M}]$, together with left and right A-actions given by the $[H,\mathscr{M}]$-morphisms

$$l_{V\otimes_A W} = (l_V \otimes_A \mathrm{id}_W) \circ ((\rho_A \otimes \rho_V) \otimes_A \rho_W)(\phi^{-1}) \ , \tag{2.3.18a}$$

$$r_{V\otimes_A W} = l_{V\otimes_A W} \circ (\rho_A \otimes (\rho_V \otimes_A \rho_W))(\mathrm{id}_H \otimes \Delta)(R_{21}) \circ \sigma \ . \tag{2.3.18b}$$

For notational convenience we refer to the denominator of the quotient in (2.3.17) by

$$N_{\rho_V,\rho_W} := \mathrm{Im}\big(r_{\rho_V} \otimes \mathrm{id} - (\mathrm{id} \otimes l_{\rho_W}) \circ \Phi_{\rho_V,\rho_A,\rho_W}\big) \ . \tag{2.3.19}$$

As a consequence of the equivalence relation in (2.3.17), one has the identity

$$r_V \otimes_A \mathrm{id}_W = (\mathrm{id}_V \otimes_A l_W) \circ (\rho_V \otimes_A (\rho_A \otimes \rho_W)(\phi) \ , \tag{2.3.20}$$

for any ρ_V, ρ_W in H-Bimod$(A)^{\mathrm{sym}}$.

By the same calculation as in Section 2.1 given any morphism $\big(f : \rho_V \Rightarrow \rho_X, g : \rho_W \Rightarrow \rho_Y\big)$ in H-Bimod$(A)^{\mathrm{sym}} \times H$-Bimod$(A)^{\mathrm{sym}}$, $f \otimes_A g : \rho_V \otimes_A \rho_W \Rightarrow \rho_X \otimes_A \rho_Y$ with single component defined as in Section 2.1 is an H-Bimod$(A)^{\mathrm{sym}}$-morphism which we define by setting

$$(f \otimes_A g) \circ \pi_{\rho_V,\rho_W} = \pi_{\rho_X,\rho_Y} \circ (f \otimes g) \ . \tag{2.3.21}$$

The properties of a quasi-bialgebra (2.1.98c) and symmetric bimodule (2.3.6) ensure that the associator Φ in $[H,\mathscr{M}]$ descends to the quotients and thereby induces an associator $\Phi^A : \otimes_A \circ \big(\otimes_A \times \mathrm{id}_{H\text{-Bimod}(A)^{\mathrm{sym}}}\big) \Rightarrow \otimes_A \circ \big(\mathrm{id}_{H\text{-Bimod}(A)^{\mathrm{sym}}} \times \otimes_A\big)$ with (ρ_V, ρ_W, ρ_X)-component

$$\Phi^A_{\rho_V,\rho_W,\rho_X} : (\rho_V \otimes_A \rho_W) \otimes_A \rho_X \Longrightarrow \rho_V \otimes_A (\rho_W \otimes_A \rho_X) \ ,$$

in H-Bimod$(A)^{\mathrm{sym}}$.

Again, declaring ρ_A (regarded as the one-dimensional free A-bimodule, cf. Example 2.3.4) as the unit object in H-Bimod$(A)^{\mathrm{sym}}$, the unitors $\varrho^A : - \otimes_A \rho_A \Rightarrow$

$\mathrm{id}_{H\text{-Bimod}(A)^{\mathrm{sym}}}$ and $\lambda^A : \rho_A \otimes_A - \Rightarrow \mathrm{id}_{H\text{-Bimod}(A)^{\mathrm{sym}}}$ with ρ_V-components

$$\lambda^A_{\rho_V} : \rho_A \otimes_A \rho_V \Longrightarrow \rho_V \ , \tag{2.3.22a}$$

$$\varrho^A_{\rho_V} : \rho_V \otimes_A \rho_A \Longrightarrow \rho_V \ , \tag{2.3.22b}$$

in H-Bimod$(A)^{\mathrm{sym}}$ are defined as in Section 2.1 by using the fact that $l_V : \rho_A \otimes \rho_V \Rightarrow \rho_V$ and $r_V : \rho_V \otimes \rho_A \Rightarrow \rho_V$ are H-Bimod$(A)^{\mathrm{sym}}$-morphisms that descend to the quotients (the properties in Lemma 2.1.20 hold when interpreted in the category $[H, \mathscr{M}]$). In summary, this shows

Proposition 2.3.6. *For any quasitriangular quasi-Hopf algebra H and any braided commutative algebra ρ_A in $[H, \mathscr{M}]$, the category H-Bimod$(A)^{\mathrm{sym}}$ of symmetric ρ_A-bimodules in $[H, \mathscr{M}]$ is a monoidal category with monoidal functor $\otimes_A$ (cf. (2.3.16) and (2.3.17)), associator Φ^A (cf. (2.3.22a)), unit object ρ_A (regarded as the one-dimensional free ρ_A-bimodule, cf. Example 2.3.4), and unitors λ^A and ρ^A (cf. (2.3.22)).*

2.3.4 Cochain twisting the monoidal structure

The monoidal category developed in Proposition 2.3.6 behaves nicely under cochain twisting.

Theorem 2.3.7. *If H is a quasitriangular quasi-Hopf algebra, ρ_A is any braided commutative algebra in $[H, \mathscr{M}]$ and $F \in H \otimes H$ is any cochain twist based on H, then the equivalence of categories in Proposition 2.3.5 can be promoted to an equivalence between the monoidal categories H-Bimod$(A)^{\mathrm{sym}}$ and H_F-Bimod$(A_F)^{\mathrm{sym}}$. Explicitly, the coherence maps are given by the H_F-Bimod$(A_F)^{\mathrm{sym}}$-isomorphisms*

$$\varphi^A_{\rho_V, \rho_W} : \mathcal{F}(\rho_V) \otimes_{A_F} \mathcal{F}(\rho_W) \Longrightarrow \mathcal{F}\big(\rho_V \otimes_A \rho_W\big) \ ,$$

with single component

$$\varphi^A_{V,W} := (\rho_V \otimes_A \rho_W)(F^{-1}) \ ,$$

*for any two objects ρ_V, ρ_W in H-*Bimod$(A)^{\mathrm{sym}}$*, and*

$$\psi^A : \rho_{A_F} \Longrightarrow \mathcal{F}(\rho_A) \ , \tag{2.3.23a}$$

with single component the identity on A.

Proof. The only non-trivial step is to prove that $\varphi^A_{V,W}$ is well defined, which amounts to proving that the singe component of the H_F-Bimod$(A_F)^{\mathrm{sym}}$-morphism $\mathcal{F}(\pi_{\rho_V,\rho_W})\circ \varphi_{\rho_V,\rho_W}$ descends to the quotient by $N^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W)}$ (cf. (2.3.19)). We have

$$\begin{aligned}
&\mathcal{F}(\pi_{V,W}) \circ \varphi_{V,W} \circ (r_{V_F} \otimes_F \mathrm{id}_W) \\
&\qquad = (r_V \otimes_A \mathrm{id}_W) \circ ((\rho_V \otimes \rho_A) \otimes_A \rho_W)\big((\Delta \otimes 1)(F^{-1}) \cdot (F^{-1} \otimes 1)\big) \\
&\qquad = (\mathrm{id}_V \otimes_A l_W) \circ (\rho_V \otimes (\rho_A \otimes \rho_W))\big(\phi \cdot (\Delta \otimes 1)(F^{-1}) \cdot (F^{-1} \otimes 1)\big) \\
&\qquad = (\mathrm{id}_V \otimes_A l_W) \circ (\rho_V \otimes (\rho_A \otimes \rho_W))\big((1 \otimes \Delta)(F^{-1}) \cdot (1 \otimes F^{-1}) \cdot \phi_F\big) \\
&\qquad = \mathcal{F}(\pi_{V,W}) \circ \varphi_{V,W} \circ (\mathrm{id}_V \otimes_F l_{W_F}) \circ (\rho_V \otimes (\rho_A \otimes \rho_W))(\phi_F) \ .
\end{aligned} \tag{2.3.24}$$

In the first equality we have used (2.2.25) and (2.3.11), in the second equality we have used (2.3.20) and in the third equality we have used (2.1.109). This implies that $\mathcal{F}(\pi_{\rho_V,\rho_W}) \circ \varphi_{\rho_V,\rho_W}$ vanishes on $N^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W)}$ and hence it descends to the desired coherence map $\varphi^A_{\rho_V,\rho_W}$ on the quotient $\mathcal{F}(\rho_V) \otimes_{A_F} \mathcal{F}(\rho_W) = \mathcal{F}(\rho_V) \otimes_F \mathcal{F}(\rho_W)/N^F_{\mathcal{F}(\rho_V),\mathcal{F}(\rho_W)}$. By a similar calculation to above using (2.3.18) it is straightforward to check that $\varphi^A_{\rho_V,\rho_W}$ is an H_F-Bimod$(A_F)^{\mathrm{sym}}$-isomorphism and that the analogues of the coherence diagrams in (2.2.27) commute. □

2.3.5 The internal hom-structure

Interpreting the theory of Subsection 2.1.5 in the closed braided monoidal category $[H, \mathscr{M}]$ makes sense.

We fix ρ_A to be a commutative algebra object in $[H, \mathscr{M}]$. The properties of a quasitriangular quasi-Hopf algebra are such that the statement of Lemma 2.1.23 holds true in $[H, \mathscr{M}]$:

Lemma 2.3.8. *For any object ρ_V in H-*Bimod$(A)^{\mathrm{sym}}$ *the $[H,\mathscr{M}]$-morphism*

$$\widehat{l}_V := \zeta_{\rho_A,\rho_V,\rho_V}(l_V) : \rho_A \Longrightarrow \mathrm{end}(\rho_V) \tag{2.3.25}$$

*is an H-*Alg*-morphism with respect to the algebra structure on* end(ρ_V) *described in Example 2.2.20.*

Proof. Acting with $\widehat{l}_V$ on the H-invariant unit element $1_A = \eta_A(1) \in A$ and using the explicit expression for the currying map (2.2.36) together with $(\epsilon\otimes 1\otimes 1)(\phi) = 1\otimes 1$ we obtain

$$\widehat{l}_V(1_A) = \zeta_{A,V,V}(l_V)(1_A) = \rho_V(\beta) = 1_{\mathrm{end}(V)} \ . \tag{2.3.26}$$

To show that $\widehat{l}_V$ preserves the product, we notice that from Proposition 2.2.13 (i) we have that

$$\mathrm{ev}\circ(\widehat{l}_V\otimes \mathrm{id}) = l_V \ . \tag{2.3.27}$$

Using this and (2.2.88) we have

$$\begin{aligned}
\bullet\circ(\widehat{l}_V\otimes\widehat{l}_V) &= \zeta(\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi)\circ(\widehat{l}_V\otimes\widehat{l}_V)\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ(\widehat{l}_V\otimes\widehat{l}_V\otimes\mathrm{id})\circ(\rho_A\otimes(\rho_A\otimes\rho_V))(\phi)\circ\zeta(\mathrm{id})\\
&= l_V\circ(\mathrm{id}_A\otimes l_V)\circ(\rho_A\otimes(\rho_A\otimes\rho_V))(\phi)\circ\zeta(\mathrm{id})\\
&= l_V\circ(\mu_A\otimes\mathrm{id}_V)\circ\zeta(\mathrm{id})\\
&= \widehat{l}_V\circ\mu_A \ .
\end{aligned} \tag{2.3.28}$$

The second equality follows from (2.2.88) and the H-equivariance of $\widehat{l}_V$. The third equality follows from (2.3.27). The fourth equality follows from (2.3.6a) and the last equality follows from (2.2.88) and the H-equivariance of μ_A. Hence $\widehat{l}_V$ is an H-Alg-morphism. □

Hence the $[H,\mathscr{M}]$-morphisms defined by

$$l_{\hom(V,W)} := \bullet_{\rho_V,\rho_W,\rho_W} \circ \big(\widehat{l}_W \otimes \mathrm{id}_{\hom(\rho_V,\rho_W)}\big) \ , \tag{2.3.29a}$$

$$r_{\hom(V,W)} = \bullet_{\rho_V,\rho_V,\rho_W} \circ (\mathrm{id}_{\hom(\rho_V,\rho_W)} \otimes \widehat{l}_V) \ , \tag{2.3.29b}$$

equip the object $\hom(\rho_V,\rho_W) \in [H,\mathscr{M}]$, where $(\rho_V,\rho_W) \in \big(H\text{-Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times H\text{-Bimod}(A)^{\mathrm{sym}}$, obtained by using the functor

$$\hom \circ \big(\mathsf{Forget}^{\mathrm{op}} \times \mathsf{Forget}\big) : \big(H\text{-Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times H\text{-Bimod}(A)^{\mathrm{sym}} \longrightarrow [H,\mathscr{M}] \ , \tag{2.3.30}$$

with the forgetful functor $\mathsf{Forget} : H\text{-Bimod}(A)^{\mathrm{sym}} \to [H,\mathscr{M}]$, with the structure of an ρ_A-bimodule in $[H,\mathscr{M}]$ (cf. Definition 2.1.16). The result of Proposition 2.2.13 (iii) provides the weak associativity of the ρ_A-actions.

Before considering the properties of the functor (2.3.30) on $H\text{-Bimod}(A)^{\mathrm{sym}}$-morphisms we need to consider Lemma 2.1.24 in the context of $[H,\mathscr{M}]$. The correct statement of this Lemma involves the use of the map ϑ which sends $[H,\mathscr{M}]$-morphisms into H-invariant internal homomorphisms in $[H,\mathscr{M}]$ (in the context of $\mathscr{M}$ this is the identity):

Lemma 2.3.9. *Given any two objects ρ_V and ρ_W in $H\text{-Bimod}(A)^{\mathrm{sym}}$ and any $H\text{-Bimod}(A)^{\mathrm{sym}}$-morphism $f : \rho_V \Rightarrow \rho_W$, we have*

$$[\vartheta_{V,W}(f), a] = 0 \ , \tag{2.3.31}$$

for any $a \in A$ where $\vartheta_{V,W}$ is defined in Proposition 2.2.38.

Proof. We have

$$
\begin{aligned}
r_{\mathrm{hom}(V,W)}(\vartheta(f)\otimes a) &= \vartheta(f)\bullet \widehat{l}_V(a)\\
&= f\circ \widehat{l}_V(a)\\
&= f\circ l_V\circ \zeta(\mathrm{id})(a)\\
&= l_W\circ (\mathrm{id}_A\otimes f)\circ \zeta(\mathrm{id})(a)\\
&= l_W\circ \zeta(\mathrm{id})(a)\circ f\\
&= \widehat{l}_W(a)\circ f\\
&= \widehat{l}_W(a)\bullet \vartheta(f)\\
&= l_{\mathrm{hom}(V,W)}\circ \tau_{\mathrm{hom}(V,W),A}(\vartheta(f)\otimes a)\ .
\end{aligned}
\tag{2.3.32}
$$

The second and penultimate steps follow from Proposition 2.2.38 (iii). The third step follows from (2.2.88). The fourth step follows from left ρ_A-linearity of f (2.3.2) and the fifth step follows from (2.2.88) and also H-equivariance of f. The last step follows from the H-invariance of $\vartheta(f)$ and the counitality of the R-matrix (2.1.105). □

Now, using the functor (2.3.30), the $[H,\mathscr{M}]$-morphism $\mathrm{hom}(f^{\mathrm{op}},g)$ with $\big(f^{\mathrm{op}}:\rho_V\Rightarrow\rho_X, g:\rho_W\Rightarrow\rho_Y\big)$ in $\big(H\text{-}\mathrm{Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}}\times H\text{-}\mathrm{Bimod}(A)^{\mathrm{sym}}$ preserves the left and right ρ_A-actions in (2.3.29): We have

$$
\begin{aligned}
\mathrm{hom}(f^{\mathrm{op}},g)\big(l_{\mathrm{hom}(V,W)}(a\otimes L)\big) &= g\circ\big(\widehat{l}_W(a)\bullet_{V,W,W} L\big)\circ f\\
&= \big(\vartheta_{W,Y}(g)\bullet_{W,W,Y}\widehat{l}_W(a)\big)\bullet_{X,W,Y}\big(L\bullet_{X,V,W}\vartheta_{X,V}(f)\big)\\
&= \big(\widehat{l}_Y(a)\bullet_{W,Y,Y}\vartheta_{W,Y}(g)\big)\bullet_{X,W,Y}\big(L\bullet_{X,V,W}\vartheta_{X,V}(f)\big)\\
&= \widehat{l}_Y(a)\bullet_{X,Y,Y}\mathrm{hom}(f^{\mathrm{op}},g)\big(L\big)\\
&= l_{\mathrm{hom}(X,Y)}\big(a\otimes \mathrm{hom}(f^{\mathrm{op}},g)\big(L\big)\big)\ ,
\end{aligned}
\tag{2.3.33}
$$

for all $a\in A$ and $L\in\mathrm{hom}(V,W)$. In the third equality we have used Lemma 2.3.9, while the second and fourth equalities follow from H-invariance and the properties of $\vartheta_{W,Y}(g)$ and $\vartheta_{X,V}(f)$, cf. Proposition 2.2.38 (iii). By a similar argument, one shows that $\hom(f^{\mathrm{op}}, g)$ also preserves the right ρ_A-action in (2.3.29).

In order to restrict the target category of the functor (2.3.30) to H-Bimod$(A)^{\mathrm{sym}}$ we consider the $[H, \mathscr{M}]$-morphism

$$[\,\cdot\,,\,\cdot\,] := r_{\hom(V,W)} - l_{\hom(V,W)} \circ \tau_{\hom(V,W),A} \ , \tag{2.3.34}$$

Using this bracket, for any two objects ρ_V, ρ_W in H-Bimod$(A)^{\mathrm{sym}}$ Definition 2.1.25 becomes

Definition 2.3.10. We define an object $\hom_A(\rho_V, \rho_W)$ in $[H, \mathscr{M}]$ by the equalizer

$$\begin{tikzcd} \hom_A(\rho_V,\rho_W) \arrow[r, Rightarrow] & \hom(\rho_V,\rho_W) \arrow[r, shift left, "\zeta([\,\cdot\,,\,\cdot\,])"] \arrow[r, shift right, "0"'] & \hom(\rho_A, \hom(\rho_V,\rho_W)) \end{tikzcd} \tag{2.3.35}$$

in $[H, \mathscr{M}]$. This equalizer can be realized explicitly in terms of the $[H, \mathscr{M}]$-subobject

$$\hom_A(\rho_V, \rho_W) := \mathrm{Ker}\big(\zeta([\,\cdot\,,\,\cdot\,])\big) \subseteq \hom(\rho_V, \rho_W) \tag{2.3.36}$$

of the internal hom-object $\hom(V, W)$ in $[H, \mathscr{M}]$.

Furthermore the result of Lemma 2.1.26 holds by the same calculations and we have the important

Lemma 2.3.11. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$ and ρ_V, ρ_W be any two objects in H-Bimod$(A)^{\mathrm{sym}}$. An $[H, \mathscr{M}]$-subobject $\rho_U \subseteq \hom(\rho_V, \rho_W)$ is an $[H, \mathscr{M}]$-subobject of $\hom_A(\rho_V, \rho_W)$ if and only if*

$$[L, a] = 0 \ , \tag{2.3.37}$$

for all $L \in U$ and $a \in A$.

It follows that

$$r_{\hom_A(V,W)} = l_{\hom_A(V,W)} \circ \tau_{\hom_A(V,W),A} \ . \tag{2.3.38}$$

That is when restricted to $\hom_A(\rho_V,\rho_W)$, the left and right ρ_A-actions agree up to the braiding. Hence equations (2.3.29) endow the object $\hom_A(\rho_V,\rho_W)$ in $[H,\mathscr{M}]$ in (2.3.36) with the structure of an object in $H\text{-Bimod}(A)^{\mathrm{sym}}$.

Furthermore, for any $\big(H\text{-Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}}\times H\text{-Bimod}(A)^{\mathrm{sym}}$-morphism $(f^{\mathrm{op}}:\rho_V\Rightarrow\rho_X, g:\rho_W\Rightarrow\rho_Y)$ we have that $\hom_A(f^{\mathrm{op}},g)(L)$ is an $H\text{-Bimod}(A)^{\mathrm{sym}}$-morphism for any $L\in\hom_A(V,W)$: First we notice that the target is correct since

$$\begin{aligned}\big[\hom_A(f^{\mathrm{op}},g)(L),a\big] &= \big[g\circ L\circ f,a\big]\\ &= g\circ L\circ[\vartheta(f),a] \,+\, 2\,g\circ\big[L,a\big]\circ f \,+\, [\vartheta(g),a]\circ L\circ f\\ &= 0\ ,\end{aligned}\tag{2.3.39}$$

where we have used the mapping of morphisms to H-invariant internal homomorphisms given in Proposition 2.2.38, the quasi biderivation property given in item (iii) of Proposition 2.1.13, the composition property given in item (ii) of Proposition 2.2.38 and the result of Lemma 2.3.9. That $\hom_A(f^{\mathrm{op}},g)(L)$ preserves the left and right ρ_A-actions follows from the calculation (2.3.33) and the general result that morphisms preserving the left action of symmetric bimodules automatically also preserve the right action (cf. Remark 2.3.3).

So the assignment of the objects $\hom_A(\rho_V,\rho_W)$ in $H\text{-Bimod}(A)^{\mathrm{sym}}$ is functorial and we denote the corresponding functor by

$$\hom_A:\big(H\text{-Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}}\times H\text{-Bimod}(A)^{\mathrm{sym}}\longrightarrow H\text{-Bimod}(A)^{\mathrm{sym}}\ .\tag{2.3.40}$$

Finally, we show that (2.3.40) is an internal hom-functor in $H\text{-Bimod}(A)^{\mathrm{sym}}$.

Proposition 2.3.12. *The braided monoidal category $H\text{-Bimod}(A)^{\mathrm{sym}}$ is closed: There is a natural bijection*

$$\zeta^A:\mathrm{Hom}_{H\text{-Bimod}(A)^{\mathrm{sym}}}(-\otimes_A -,-)\Longrightarrow\mathrm{Hom}_{H\text{-Bimod}(A)^{\mathrm{sym}}}(-,\hom_A(-,-))\ ,\tag{2.3.41}$$

with components given by

$$\zeta^A(f) := f \circ \big(\rho_V(\phi^{(-1)}) \otimes_A \rho_W(\phi^{(-2)}\,\beta\, S(\phi^{(-3)}))\big) : \rho_V \Longrightarrow \mathrm{hom}_A(\rho_W, \rho_X) \;, \tag{2.3.42}$$

*for all H-*Bimod$(A)^{\mathrm{sym}}$*-morphisms $f : \rho_V \otimes_A \rho_W \Rightarrow \rho_X$. The components of its inverse are*

$$(\zeta^A)^{-1}(g) := \rho_X(\phi^{(1)}) \circ g(-) \circ \rho_W(S(\phi^{(2)})\alpha\phi^{(3)}) : \rho_V \otimes_A \rho_W \Longrightarrow \rho_X \;, \tag{2.3.43}$$

*for all H-*Bimod$(A)^{\mathrm{sym}}$*-morphisms $g : \rho_V \Rightarrow \mathrm{hom}_A(\rho_W, \rho_X)$.*

Proof. We show that (1) the image of $\zeta^A(f)$ is contained in $\mathrm{hom}_A(\rho_W, \rho_X)$ for all H–Bimod$(A)^{\mathrm{sym}}$-morphisms $f : \rho_V \otimes_A \rho_W \Rightarrow \rho_X$, (2) if $f : \rho_V \otimes_A \rho_W \Rightarrow \rho_X$ is an H–Bimod$(A)^{\mathrm{sym}}$-morphism, then $\zeta^A_{V,W,X}(f) : \rho_V \Rightarrow \mathrm{hom}(\rho_W, \rho_X)$ is an H–Bimod$(A)^{\mathrm{sym}}$-morphism, (3) $(\zeta^A_{V,W,X})^{-1}(g)$ is well-defined, and (4) if $g : \rho_V \Rightarrow \mathrm{hom}_A(\rho_W, \rho_X)$ is an H–Bimod$(A)^{\mathrm{sym}}$-morphism, then $(\zeta^A)^{-1}(g) : \rho_V \otimes_A \rho_W \Rightarrow \rho_X$ is an H–Bimod$(A)^{\mathrm{sym}}$-morphism.

For (1) we have

$$\begin{aligned}
r_{\mathrm{hom}(W,X)} \circ (\zeta^A(f)(-) \otimes \mathrm{id}_A) &= \bullet \circ (\zeta^A(f)(-) \otimes \widehat{l}_W(-)) \\
&= \mathrm{ev}\big(\zeta^A(f)(-) \otimes \mathrm{ev}(\widehat{l}_W(-) \otimes -\big) \circ \Phi_{V,A,W} \circ \zeta(\mathrm{id}) \\
&= f \circ (\mathrm{id}_V \otimes_A l_W) \circ \Phi_{V,A,W} \circ \zeta^A(\mathrm{id}) \\
&= \zeta^A(f) \circ (r_V \otimes_A \mathrm{id}_W) \;.
\end{aligned} \tag{2.3.44}$$

The first equality follows from (2.3.29b), the second equality follows from the definition of $\bullet$ together with (2.2.88), the third equality follows from (2.3.27) and Proposition 2.2.13 (i) and the fourth equality follows from (2.3.20) and (2.2.88). On the other hand, by a similar calculation we have

$$
\begin{aligned}
& l_{\hom(W,X)} \circ \tau_{\hom(W,X),A} \circ (\zeta^A(f)(-) \otimes \mathrm{id}_A) \\
& = \bullet \circ (\widehat{l}_X(-) \otimes \zeta^A(f)(-)) \circ (\tau_{V,A} \otimes_A \mathrm{id}_W) \\
& = l_X \circ (\mathrm{id}_A \otimes f) \circ \Phi_{A,V,W} \circ (\tau_{V,A} \otimes_A \mathrm{id}_W) \circ \zeta^A(\mathrm{id}) \\
& = f \circ l_{V \otimes_A W} \circ \Phi_{A,V,W} \circ (\tau_{V,A} \otimes_A \mathrm{id}_W) \circ \zeta^A(\mathrm{id}) \\
& = f \circ (l_V \otimes_A \mathrm{id}_W) \circ (\tau_{V,A} \otimes_A \mathrm{id}_W) \circ \zeta^A(\mathrm{id}) \\
& = \zeta^A(f) \circ (r_V \otimes_A \mathrm{id}_W) \ .
\end{aligned} \tag{2.3.45}
$$

The first equality follows from the H-equivariance of $\widehat{l}_X$ and $\zeta^A(f)$. The third equality follows from the left ρ_A-linearity of f. The fourth equality follows from (2.3.18a) and the fifth equality follows from the definition of the right ρ_A-action in a symmetric bimodule. This shows that

$$
(r_{\hom(W,X)} - l_{\hom(W,X)} \circ \tau_{\hom(W,X),A}) \circ (\zeta^A(f)(-) \otimes \mathrm{id}_A) = 0 \ . \tag{2.3.46}
$$

Due to Lemma 2.3.11 $\zeta^A(f)(-) \in \hom_A(\rho_W, \rho_X)$.

For (2) we have

$$
\begin{aligned}
\zeta^A(f) \circ l_V &= f \circ \zeta^A(\mathrm{id}) \circ (l_V \otimes_A \mathrm{id}_W) \\
&= f \circ (l_V \otimes_A \mathrm{id}_W) \circ \zeta^A(\mathrm{id}) \\
&= f \circ l_{V \otimes_A W} \circ \Phi_{A,V,W} \circ \zeta^A(\mathrm{id}) \\
&= l_X \circ (\mathrm{id}_A \otimes f) \circ \Phi_{A,V,W} \circ \zeta^A(\mathrm{id}) \ .
\end{aligned} \tag{2.3.47}
$$

The first equality follows from (2.2.88), the second equality follows from the H-equivariance of l_V, the third equality follows from (2.3.18a) and the fourth equality follows from the H-equivariance of f. On the other hand we have by a similar calculation to that for (1) that

$$l_{\hom_A(W,X)} \circ (\mathrm{id}_A \otimes \zeta^A(f)(-)) = \bullet \circ (\widehat{l}_X(-) \otimes \zeta^A(f)(-))$$

$$= l_X \circ (\mathrm{id}_A \otimes f) \circ \Phi_{A,V,W} \circ \zeta^A(\mathrm{id}) \ . \tag{2.3.48}$$

The first equality follows from (2.3.29a), the second equality follows from the definition of $\bullet$ together with (2.2.88) and the third equality follows from (2.3.27) and Proposition 2.2.13 (i). This shows that $\zeta^A(f)$ is left ρ_A-linear, i.e.

$$\zeta^A(f) \circ l_V = l_{\hom(W,X)} \circ (\mathrm{id}_A \otimes \zeta^A(f)(-)) \ . \tag{2.3.49}$$

Right ρ_A-linearity follows by the fact that ρ_V and $\hom_A(\rho_W, \rho_X)$ are symmetric bimodules and the general result in Remark 2.3.3.

In the following two proofs we suppress the elements of H appearing in the definition of the inverse currying (2.3.43).

For (3) we have

$$\begin{aligned}
(\zeta^A)^{-1}(g) \circ (r_V \otimes_A \mathrm{id}_W) &= \rho_X() \circ g \circ r_V \circ \rho_W() \\
&= \rho_X() \circ r_{\hom(W,X)} \circ (g(-) \otimes_A \mathrm{id}_A) \circ \rho_W() \\
&= \rho_X() \circ \bullet \circ (g(-) \otimes_A \widehat{l}_W(-)) \circ \rho_W() \\
&= \rho_X() \circ \mathrm{ev} \circ (g(-) \otimes_A l_W) \circ \rho_W() \circ \Phi_{V,A,W} \circ \zeta(\mathrm{id}) \\
&= \rho_X() \circ g(-) \circ \rho_W() \circ (\mathrm{id}_V \otimes_A l_W) \circ \Phi_{V,A,W} \\
&= (\zeta^A)^{-1}(g) \circ (\mathrm{id}_V \otimes_A l_W) \circ \Phi_{V,A,W} \ .
\end{aligned} \tag{2.3.50}$$

The second equality follows from the right ρ_A-linearity of g, the fourth equality follows from the definition of $\bullet$, and the fifth equality follows from the H-equivariance of l_W and $\mathrm{ev} = \zeta^{-1}(\mathrm{id})$ together with (2.2.88). So $(\zeta^A)^{-1}(g)$ is well defined on equivalence classes of the monoidal functor $\otimes_A$ in H-$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

(4) follows from the calculation

$$
\begin{aligned}
(\zeta^A)^{-1}(g) \circ l_{V\otimes_A W} &= \rho_X() \circ g(-) \circ \rho_W() \circ l_{V\otimes_A W} \\
&= \rho_X() \circ g(-) \circ \rho_W() \circ (l_V \otimes_A \mathrm{id}_W) \circ \Phi^{-1}_{A,V,W} \\
&= \rho_X() \circ g \circ l_V \circ \rho_W() \circ \Phi^{-1}_{A,V,W} \\
&= \rho_X() \circ l_{\hom(W,X)} \circ (\mathrm{id}_A \otimes g(-)) \circ \rho_W() \circ \Phi^{-1}_{A,V,W} \\
&= \rho_X() \circ \bullet \circ (\widehat{l}_X(-) \otimes g(-)) \circ \rho_W() \circ \Phi^{-1}_{A,V,W} \\
&= \rho_X() \circ l_X \circ \big(\mathrm{id}_A \otimes \mathrm{ev}(g(-) \otimes \mathrm{id}_W)\big) \circ \rho_W() \circ \zeta(\mathrm{id}) \\
&= l_X \circ \rho_{A\otimes X}() \circ \big(\mathrm{id}_A \otimes \mathrm{ev}(g(-) \otimes \mathrm{id}_W)\big) \circ \rho_W() \circ \zeta(\mathrm{id}) \\
&= l_X \circ \big(\mathrm{id}_A \otimes \rho_X() \circ g(-) \circ \rho_W()\big) \\
&= l_X \circ \big(\mathrm{id}_A \otimes (\zeta^A)^{-1}(g)\big) \ . \qquad (2.3.51)
\end{aligned}
$$

The second equality follows from (2.3.18a), the fourth equality follows from the left ρ_A-linearity of g, the sixth equality follows from the definition of $\bullet$ and (2.3.27), the seventh equality follows from the H-equivariance of l_X and the eighth equality follows from the fact that $\rho_A() \circ \zeta(\mathrm{id}) \circ \rho_W() = \zeta^{-1}(\zeta(\mathrm{id})) = \mathrm{id}$ and $\mathrm{ev} = \zeta^{-1}(\mathrm{id})$ together with (2.2.88). This proves the left ρ_A-linearity of $(\zeta^A)^{-1}(g)$. The right ρ_A-linearity follows by Remark 2.3.3.

Remark 2.3.13. We notice that this calculation also implies that the $[H,\mathscr{M}]$-morphism

$$\mathrm{ev} := \zeta^{-1}(\mathrm{id}_{\hom(\rho_V,\rho_W)}) \qquad (2.3.52)$$

is left ρ_A-linear for $\rho_V, \rho_W \in H\text{–}\mathrm{Bimod}(A)$ and also right ρ_A-linear for $\rho_V, \rho_W \in H\text{–}\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Naturality of ζ^A and the fact that $(\zeta^A)^{-1}$ is the inverse of ζ is easily seen and completely analogous to (2.1.17).

□

2.3.6 Cochain twisting the internal hom-structure

Given any cochain twist $F = F^{(1)} \otimes F^{(2)} \in H \otimes H$ based on H with inverse $F^{-1} = F^{(-1)} \otimes F^{(-2)} \in H \otimes H$, Theorem 2.3.7 implies that the monoidal categories H-Bimod$(A)^{\mathrm{sym}}$ and H_F-Bimod$(A_F)^{\mathrm{sym}}$ are equivalent; recall that we have denoted the corresponding monoidal functor by $\mathcal{F} : H\text{-Bimod}(A)^{\mathrm{sym}} \to H_F\text{-Bimod}(A_F)^{\mathrm{sym}}$. We now prove that this equivalence also respects the internal hom-functors.

First we require the following technical

Lemma 2.3.14. *Let $F \in H \otimes H$ be any cochain twist and ρ_V any object in H–*Bimod$(A)^{\mathrm{sym}}$. *Denoting by $\widehat{l}_{V_F} : \rho_{A_F} \Rightarrow \mathrm{end}_F(\mathcal{F}(\rho_V))$ the H_F–*Alg-*morphism $\zeta_F(l_{V_F})$ (cf.* (2.3.8)*). Then the diagram*

$$\begin{tikzcd} \mathcal{F}(\rho_A) \arrow[r, Rightarrow, "\widehat{l}_{V_F}"] \arrow[dr, Rightarrow, "\mathcal{F}(\widehat{l}_V)"'] & \rho_{\mathrm{end}_F(\mathcal{F}(V))} \arrow[d, Rightarrow, "\gamma"] \\ & \mathcal{F}\big(\rho_{\mathrm{end}(V)}\big) \end{tikzcd} \tag{2.3.53}$$

commutes. That is

$$\mathcal{F}(\widehat{l}_V) = \gamma \circ \widehat{l}_{V_F} \ . \tag{2.3.54a}$$

As a consequence the bracket (2.3.34) *is twisted to*

$$[\,\cdot\,,\,\cdot\,]_F = \gamma^{-1} \circ \mathcal{F}\big([\,\cdot\,,\,\cdot\,]\big) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \ . \tag{2.3.54b}$$

Proof. We use (2.2.101) and (2.3.10) and (2.2.88) to show that

$$\begin{aligned} \gamma \circ \widehat{l}_{V_F} &= \gamma \circ l_{V_F} \circ \zeta_F(\mathrm{id}) \\ &= \gamma \circ \mathcal{F}(l_V) \circ (\varphi \otimes \mathrm{id}) \circ (\varphi^{-1} \otimes \mathrm{id}) \circ \gamma^{-1} \circ \zeta(\mathrm{id}) \\ &= \mathcal{F}(\zeta(l_V)) \\ &= \mathcal{F}(\widehat{l}_V) \ . \end{aligned} \tag{2.3.55}$$

Equation (2.3.54b) now follows directly from (2.2.108b), (2.3.34) and (2.3.54a). □

Proposition 2.3.15. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$ and let F be any cochain twisting element based on H. Then the coherence map $\gamma : \hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \Rightarrow \mathcal{F}(\hom(\rho_V, \rho_W))$ restricts to an $[H_F, \mathscr{M}]$-isomorphism*

$$\gamma : \hom_{A_F}(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \Longrightarrow \mathcal{F}(\hom_A(\rho_V, \rho_W)) \ . \tag{2.3.56}$$

Proof. The braided closed monoidal functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ is an equivalence of categories, hence it preserves all limits and colimits. It then follows that $\mathcal{F}(\hom_A(\rho_V, \rho_W))$ is the equalizer of the $[H_F, \mathscr{M}]$-diagram

$$\begin{tikzcd} \mathcal{F}(\hom(\rho_V, \rho_W)) \arrow[r, Rightarrow, shift left, "\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]))"] \arrow[r, Rightarrow, shift right, "0"'] & \mathcal{F}(\hom(\rho_A, \hom(\rho_V, \rho_W))) \end{tikzcd} \ . \tag{2.3.57}$$

On the other hand, the object $\hom_{\mathcal{F}(A)}(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))$ in $[H_F, \mathscr{M}]$ is defined according to Definition 2.3.10 as the equalizer of the $[H_F, \mathscr{M}]$-diagram

$$\begin{tikzcd} \hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \arrow[r, Rightarrow, shift left, "\zeta_F([\,\cdot\,,\cdot\,]_F)"] \arrow[r, Rightarrow, shift right, "0"'] & \hom_F(\mathcal{F}(\rho_A), \hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))) \end{tikzcd} \ . \tag{2.3.58}$$

A straightforward calculation shows that the $[H_F, \mathscr{M}]$-diagrams (2.3.57) and (2.3.58) are isomorphic: The $[H_F, \mathscr{M}]$-diagram

$$\begin{tikzcd} \hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \arrow[r, Rightarrow, shift left, "\zeta_F([\,\cdot\,,\cdot\,]_F)"] \arrow[r, Rightarrow, shift right, "0"'] \arrow[dd, Rightarrow, "\gamma"'] & \hom_F(\mathcal{F}(\rho_A), \mathcal{F}(\hom(\rho_V, \rho_W))) \arrow[d, Rightarrow, "\gamma\circ(\,\cdot\,)"] \\ & \hom_F(\mathcal{F}(\rho_A), \hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))) \arrow[d, Rightarrow, "\gamma"] \\ \mathcal{F}(\hom(\rho_V, \rho_W)) \arrow[r, Rightarrow, shift left, "\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]))"] \arrow[r, Rightarrow, shift right, "0"'] & \mathcal{F}(\hom(\rho_A, \hom(\rho_V, \rho_W))) \end{tikzcd} \tag{2.3.59}$$

commutes (i.e. the diagram obtained by taking either both upper or lower horizontal arrows commutes) and the vertical arrows are all $[H_F, \mathscr{M}]$-isomorphisms. Indeed we have

$$
\begin{aligned}
&\gamma \circ (\gamma \circ (\,\cdot\,)) \circ \zeta_F([\,\cdot\,,\,\cdot\,]_F) = \gamma \circ (\gamma \circ (\,\cdot\,)) \circ [\,\cdot\,,\,\cdot\,]_F \circ \zeta_F(\mathrm{id}) \\
&\quad = \gamma \circ (\gamma \circ (\,\cdot\,)) \circ \gamma^{-1} \circ \mathcal{F}\big([\cdot,\cdot]\big) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \circ \varphi^{-1} \circ (\gamma^{-1} \otimes_F \mathrm{id}) \circ \mathcal{F}(\zeta(\mathrm{id})) \\
&\quad = \mathcal{F}\big(\zeta([\cdot,\cdot])\big) \circ \gamma \ . \qquad (2.3.60)
\end{aligned}
$$

The second equality follows from (2.3.54b) and (2.2.101) and the third equality follows from the H_F-equivariance of γ. Due to the universality of limits there is a unique isomorphism between $\hom_{\mathcal{F}(A)}(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))$ and $\mathcal{F}(\hom_A(\rho_V, \rho_W))$. The assertion now follows from the fact that the internal hom-objects in H–Bimod$(A)^{\mathrm{sym}}$ are subobjects of the internal hom-objects in $[H, \mathscr{M}]$ (cf. (2.3.35)) and hence the unique isomorphism between $\hom_{\mathcal{F}(A)}(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))$ and $\mathcal{F}(\hom_A(\rho_V, \rho_W))$ is the one induced by the isomorphism between $\hom_F(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W))$ and $\mathcal{F}(\hom(\rho_V, \rho_W))$, which is precisely γ. $\square$

We say that the twist deformation quantisation functor preserves internal homomorphisms in H–Bimod$(A)^{\mathrm{sym}}$. What we mean by this is that there is a structural isomorphism and the H-actions required to construct this isomorphism are precisely those which preserve the internal homomorphism objects in $[H, \mathscr{M}]$.

It remains to prove

Lemma 2.3.16. *For any two objects ρ_V and ρ_W in H-Bimod$(A)^{\mathrm{sym}}$ the (ρ_V, ρ_W)-component of γ (cf. (2.2.47)) induces to the H_F-Bimod$(A_F)^{\mathrm{sym}}$-isomorphism*

$$\gamma^{A}_{\rho_V,\rho_W} : \hom_{A_F}(\mathcal{F}(\rho_V), \mathcal{F}(\rho_W)) \Longrightarrow \mathcal{F}\big(\hom_A(\rho_V, \rho_W)\big) \ , \qquad (2.3.61)$$

for all objects (ρ_V, ρ_W) in $\big(H\text{–Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times H\text{–Bimod}(A)^{\mathrm{sym}}$.

Proof. We have

$$
\begin{aligned}
\gamma \circ l_{\mathrm{hom}_{A_F}(V_F, W_F)} &= \gamma \circ \bullet^F \circ (\widehat{l}_{W_F} \otimes_F \mathrm{id}) \\
&= \gamma \circ \gamma^{-1} \circ \mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma) \circ (\gamma^{-1} \circ \mathcal{F}(\widehat{l}_W) \otimes_F \mathrm{id}) \\
&= \mathcal{F}(\bullet) \circ (\mathcal{F}(\widehat{l}_W) \otimes_F \mathrm{id}) \circ \varphi \circ (\mathrm{id}_A \otimes_F \gamma) \\
&= l_{\mathcal{F}(\mathrm{hom}(V,W))} \circ (\mathrm{id}_A \otimes_F \gamma) \ . \qquad (2.3.62)
\end{aligned}
$$

The second equality follows from (2.2.104b) and (2.3.54a) and in the third equality we have used the H_F-equivariance of φ. In the final step we have used the property $l_{\mathcal{F}(\mathrm{hom}(V,W))} = l_{\mathrm{hom}(V,W)} \circ \varphi$ (cf. (2.3.8) with $\mathrm{hom}(V, W)$ in place of V). This proves left ρ_A-linearity. Right ρ_A-linearity follows from the general result (cf. Remark 2.3.3). □

In summary, we have shown

Theorem 2.3.17. *If H is a quasitriangular quasi-Hopf algebra, ρ_A is a braided commutative algebra in $[H, \mathscr{M}]$ and $F \in H \otimes H$ is any cochain twist based on H, then H-*Bimod$(A)^{\mathrm{sym}}$ *and H_F-*Bimod$(A_F)^{\mathrm{sym}}$ *are equivalent as closed monoidal categories.*

2.3.7 The braiding

Theorem 2.3.18. *Let H be a quasitriangular quasi-Hopf algebra and ρ_A any braided commutative algebra in $[H, \mathscr{M}]$. Then the braiding τ in the closed monoidal category $[H, \mathscr{M}]$ descends to a braiding τ^A in the closed monoidal category H–*Bimod$(A)^{\mathrm{sym}}$*. Explicitly, the single component of the (ρ_V, ρ_W)-component of τ^A is given by*

$$\tau^A_{\rho_V, \rho_W} : (\rho_V \otimes_A \rho_W)(R_{21}) \circ \sigma \ . \qquad (2.3.63)$$

*As a consequence, H–*Bimod$(A)^{\mathrm{sym}}$ *is a braided closed monoidal category.*

Proof. We have to show that

$$\tau^A_{\rho_V,\rho_W} = \pi_{\rho_W,\rho_V} \circ \tau_{\rho_V,\rho_W} \tag{2.3.64}$$

is a well-defined H–Bimod$(A)^{\mathrm{sym}}$-morphism, i.e. that the single component of (2.3.64) vanishes on N_{ρ_V,ρ_W} (cf. (2.3.19)) as an H–Bimod$(A)^{\mathrm{sym}}$-morphism. That $\pi_{\rho_W,\rho_V} \circ \tau_{\rho_V,\rho_W}$ is an H–Bimod$(A)^{\mathrm{sym}}$-morphism follows by standard calculations using (2.3.15a) $l_{V\otimes W} = (l_V \otimes \mathrm{id}_W) \circ \Phi^{-1}_{A,V,W}$, (2.3.20) $r_V \otimes_A \mathrm{id}_W = (\mathrm{id}_V \otimes_A l_W) \circ \Phi_{V,A,W}$, the braided symmetry of ρ_W, the bimodule properties (2.3.6) and the property of the R-matrix (2.1.103c) $(\Delta \otimes \mathrm{id}_H)(R) = \phi_{312}\, R_{13}\, \phi^{-1}_{132}\, R_{23}\, \phi_{123}$. Due to the properties of the associator and R-matrix in a quasitriangular quasi-Hopf algebra, this is shown by a calculation identical to that in (2.1.97) but inserting actions of the associator and R-matrix upon re-bracketing and flipping arguments respectively. The relevant properties are (2.1.103b) and (2.1.103c), the property of the braiding $\tau^2 = \mathrm{id}$ which implies that $r_W = l_W \circ \tau_{W,A}$, and the property (2.3.20). □

2.3.8 Cochain twisting the braiding

We now show that H–Bimod$(A)^{\mathrm{sym}}$ is a braided closed monoidal category, for any quasitriangular quasi-Hopf algebra H and any braided commutative algebra ρ_A in $[H, \mathscr{M}]$.

Let H be a quasitriangular quasi-Hopf algebra, ρ_A any braided commutative algebra in $[H, \mathscr{M}]$ and $F \in H \otimes H$ any cochain twist based on H. By Proposition 2.2.25 the twisted algebra ρ_{A_F} is a braided commutative algebra in $[H_F, \mathscr{M}]$. Recalling Theorem 2.3.17, we have an equivalence $\mathcal{F}$ of closed monoidal categories between H–Bimod$(A)^{\mathrm{sym}}$ and H_F–Bimod$(A_F)^{\mathrm{sym}}$. Since the braiding τ^A on H–Bimod$(A)^{\mathrm{sym}}$ is canonically induced by the braiding τ on $[H, \mathscr{M}]$, the same argument as in Theorem 2.2.16 shows

Theorem 2.3.19. *For any quasitriangular quasi-Hopf algebra H, any braided commutative algebra ρ_A in $[H, \mathscr{M}]$ and any cochain twist $F \in H \otimes H$, the equivalence of closed monoidal categories in Theorem 2.3.17 restricts to an equivalence of braided* *closed monoidal categories between* H–Bimod$(A)^{\mathrm{sym}}$ *and* H_F–Bimod$(A_F)^{\mathrm{sym}}$.

2.4 Summary

In this chapter we have described a categorical framework for representations of triangular quasi-Hopf algebras. We have found that the constructions in Section 2.1 for the category $\mathscr{M}$ of k-modules descend to the category $[H, \mathscr{M}]$ of representations of H if one simply inserts the R-matrix and associator of H upon flipping or re-bracketing expressions respectively and uses with some insight the structure which relates H-invariant internal homomorphisms to morphisms in the category $[H, \mathscr{M}]$. This follows from the various representations of a quasi-Hopf algebra H in terms of its structure maps, the definition of the currying bijection in $[H, \mathscr{M}]$, and the properties of the triangular quasi-Hopf algebra (H, R).

More significantly from the point of view of physics, this chapter has provided us with the notions of commutative algebras ρ_A and symmetric ρ_A-bimodules ρ_V in the category $[H, \mathscr{M}]$ and established that fixing any triangular quasi-Hopf algebra H and any commutative algebra ρ_A in $[H, \mathscr{M}]$, the category H–Bimod$(A)^{\mathrm{sym}}$ of symmetric ρ_A-bimodules in $[H, \mathscr{M}]$ is a braided monoidal category with a tensor product operation $\otimes_A$. This chapter has also provided us with the construction of an internal hom-functor $\hom_A : \big(H\text{–Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times H\text{–Bimod}(A)^{\mathrm{sym}} \to H\text{–Bimod}(A)^{\mathrm{sym}}$ together with appropriate structures with which to use internal homomorphisms in H–Bimod$(A)^{\mathrm{sym}}$ correctly as map-like objects in $[H, \mathscr{M}]$. These are the basic ingredients with which to build a theory of differential geometry on algebra objects in $[H, \mathscr{M}]$.

For any cochain twisting element $F \in H \otimes H$, we constructed in Subsection 2.2.2 a functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ between the representation category of the triangular quasi-Hopf algebra H and its cochain twist H_F. With insightful use of the structures of the triangular quasi-Hopf algebra H_F in terms of those in H, in subsequent subsections we systematically constructed coherence maps for $\mathcal{F}$ in such a way as to make $\mathcal{F}$ a braided closed monoidal functor. In other words to make the constructions in the closed braided monoidal category $[H_F, \mathscr{M}]$ structurally isomorphic to those in $[H, \mathscr{M}]$. In Subsections 2.2.11, 2.3.2 we saw that the braided closed monoidal functor $\mathcal{F} : [H, \mathscr{M}] \to [H_F, \mathscr{M}]$ induces a functor (denoted with abuse of notation by the same symbol) $\mathcal{F} : H\text{–}\mathrm{Alg}^{\mathrm{com}} \to H_F\text{–}\mathrm{Alg}^{\mathrm{com}}$ which allows us to twist quantize algebras in $[H, \mathscr{M}]$ to algebras in $[H_F, \mathscr{M}]$, and a closed braided monoidal functor $\mathcal{F} : H\text{–}\mathrm{Bimod}(A)^{\mathrm{sym}} \to H_F\text{–}\mathrm{Bimod}(A_F)^{\mathrm{sym}}$, which allows us to twist quantize bimodules together with their tensor products and internal hom-objects in $[H, \mathscr{M}]$ to bimodules together with their tensor products and internal hom-objects in $[H_F, \mathscr{M}]$. Having established this structural isomorphism between the closed braided monoidal categories $H\text{–}\mathrm{Bimod}(A)^{\mathrm{sym}}$ and $H_F\text{–}\mathrm{Bimod}(A_F)^{\mathrm{sym}}$, we can focus on developing notions of differential geometry in an arbitrary triangular quasi-Hopf representation category and restrict to that of H_F in physical applications.

In the next chapter we focus on developing notions of differential geometry in the representation category of an arbitrary triangular quasi-Hopf algebra using the constructions of this chapter. Of particular importance is the endomorphism algebra of Example 2.2.20 which together with the internal commutator (cf. Subsection 2.2.13) is promoted to a Lie algebra in Corollary 2.2.33.

Chapter 3

Geometry in quasi-Hopf representation categories

This chapter is based on [37] and contains the main contribution of this thesis.

In the previous chapter we constructed commutative algebras ρ_A and symmetric ρ_A-bimodules ρ_V in the category $[H, \mathscr{M}]$ for H a triangular quasi-Hopf algebra. We constructed a tensor product operation $\otimes_A$ for symmetric ρ_A-bimodules and an internal hom-functor $\mathrm{hom}_A : \big(H\text{–Bimod}(A)^{\mathrm{sym}}\big)^{\mathrm{op}} \times H\text{–Bimod}(A)^{\mathrm{sym}} \to H\text{–Bimod}(A)^{\mathrm{sym}}$ together with appropriate structures with which to use internal homomorphisms in H–Bimod$(A)^{\mathrm{sym}}$ correctly as map-like objects in $[H, \mathscr{M}]$. In this chapter we develop the notion of differential calculus starting from the notions of derivations of commutative algebras ρ_A and differential operators in $[H, \mathscr{M}]$. We then describe connections on symmetric ρ_A-bimodules ρ_V together with their lifts to tensor product objects and internal hom-objects in H–Bimod$(A)^{\mathrm{sym}}$. To allow for geometric entities which are not H-invariant, we build all geometric entities out of internal homomorphisms rather than morphisms. Geometric quantities which are H-invariant are then simply H-invariant internal homomorphisms (cf. Subsection 2.2.15 in Chapter 2). Although we can build up these notions directly in the category $[H_F, \mathscr{M}]$ we consider the cochain twisting of all structures. The role of twist deformation quantisation is to show existence of geometric entities on noncommutative and nonassociative spaces obtained via cochain twisting of classical manifolds.

3.1 Preliminaries

Let k be an associative and commutative ring with unit $1 \in k$. In contrast to Chapter 2, in this chapter we shall work with $\mathbb{Z}$-graded k-modules. This will have the advantage later on that naturally graded objects such as differential calculi can be described as objects in the categories we define below, and also that minus signs will be absorbed into the formalism. Since in physical examples grading will usually be bounded, we furthermore work in a category of bounded $\mathbb{Z}$-graded k-modules. (This enables us to use direct sums instead of direct products in the definitions of objects.) The goal of this section is to adapt the material developed in Chapter 2 to the graded setting.

3.1.1 $\mathbb{Z}$-graded k-modules

The category $\mathscr{M}$ (denoted with abuse of notation by the same symbol as that of ungraded objects in Chapter 2) of bounded $\mathbb{Z}$-graded k-modules is defined as follows: The objects in $\mathscr{M}$ are the bounded $\mathbb{Z}$-graded k-modules

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \ , \tag{3.1.1}$$

where the k-modules $V_n = 0$ for all but finitely many n. The morphisms in $\mathscr{M}$ are the degree preserving k-linear maps $f : V \to W$, i.e. $f(V_n) \subseteq W_n$ for all $n \in \mathbb{Z}$. For any object V in $\mathscr{M}$ there is a map

$$|\cdot| : \coprod_{n\in\mathbb{Z}} V_n \longrightarrow \mathbb{Z} \ , \tag{3.1.2}$$

which assigns to elements $v \in V_n$ their degree $|v| = n$. Elements of V_n are said to be homogeneous of degree n.

The category $\mathscr{M}$ is a closed, braided monoidal category: The constructions of Chapter 2 extend by k-linearity and distributivity of addition to the graded setting. We therefore denote all constructions with an abuse of notation by the same symbols as those in Chapter 2.

The monoidal functor $\otimes : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ is given by the $\mathbb{Z}$-graded tensor product:

$$V\otimes W := \bigoplus_{n\in\mathbb{Z}} \big(V\otimes W\big)_n := \bigoplus_{n\in\mathbb{Z}} \ \bigoplus_{m+l=n} V_m\otimes W_l \ , \tag{3.1.3}$$

for any two objects V, W in $\mathscr{M}$, where $V_m \otimes W_l$ is the usual tensor product of

k-modules. We note that $|V_m \otimes W_l| = |V_m| + |W_l|$. To any $\mathscr{M} \times \mathscr{M}$-morphism $(f : V \to V',\ g : W \to W')$ the monoidal functor assigns the $\mathscr{M}$-morphism

$$f \otimes g : V \otimes W \longrightarrow V' \otimes W' \ . \tag{3.1.4}$$

Since $\mathscr{M}$-morphisms are degree preserving there is no sign acquired upon evaluation:

$$(f \otimes g)(- \otimes -) = f(-) \otimes g(-) \ . \tag{3.1.5}$$

The unit object in $\mathscr{M}$ is given by the ring k, but regarded as a $\mathbb{Z}$-graded k-module with $k_n = 0$, for all $n \neq 0$, and $k_0 = k$. The associator and unitors in $\mathscr{M}$ are defined componentwise from those in Chapter 2 because there are decompositions

$$\big(V \otimes W\big) \otimes X = \bigoplus_{n \in \mathbb{Z}} \Big(\bigoplus_{i+j+l=n} \Big(\big(V_i \otimes W_j\big) \otimes X_l \Big) \Big) \ , \tag{3.1.6}$$

and

$$k \otimes V = \bigoplus_{n \in \mathbb{Z}} (k \otimes V_n) \ , \qquad V \otimes k = \bigoplus_{n \in \mathbb{Z}} (V_n \otimes k) \ . \tag{3.1.7}$$

The pentagon and triangle relations for monoidal categories hold because they hold on homogeneous elements and the results can be extended by the k-linearity of the associator and unitors to general elements.

We equip $\mathscr{M}$ with the braiding natural isomorphism $\sigma : \otimes \Rightarrow \otimes^{\mathrm{op}}$ defined componentwise from that in Chapter 2 by the flip functor (cf. A.3.6) but now with the additional sign

$$(-1)^{n\,m} \tag{3.1.8}$$

in front of elements of $\mathrm{flip}(\,V_n \times W_m\,)$. The hexagon relations hold on homogeneous elements and therefore on general elements by k-linearity.

The internal hom-functor

$$\hom : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathscr{M} \ , \tag{3.1.9}$$

is defined for any object (V, W) in $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$ by

$$\hom\big(V, W\big) := \bigoplus_{n\in\mathbb{Z}} \hom\big(V, W\big)_n := \bigoplus_{n\in\mathbb{Z}} \ \bigoplus_{l-m=n} \mathrm{Hom}_k\big(V_m, W_l\big) \ , \tag{3.1.10}$$

where $\mathrm{Hom}_k(V_m, W_l)$ denotes the k-module of k-linear maps between the homogeneous components V_m and W_l (note that these are not the morphisms in the graded category $\mathscr{M}$).

Remark 3.1.1. In contrast to the ungraded case, here we have that the $\mathscr{M}$-morphisms lie in a k-subspace of the internal hom-objects: To see this visually, we note that any internal hom-object $\hom(V, W)$ can be decomposed into a sum of homogeneous $\mathbb{Z}$-graded k-module maps which can be arranged in a matrix with $\mathbb{Z}$-graded k-module maps of degree δ lying on lower or upper diagonals starting either at column δ (when $\delta > 0$) or at row δ (when $\delta < 0$) respectively. Only the diagonal elements of the matrix are $\mathscr{M}$-morphisms:

$$\begin{pmatrix} \mathrm{Hom}_k\big(V_0, W_0\big) & \mathrm{Hom}_k\big(V_0, W_1\big) & \cdots & \mathrm{Hom}_k\big(V_0, W_n\big) \\ \mathrm{Hom}_k\big(V_1, W_0\big) & \mathrm{Hom}_k\big(V_1, W_1\big) & \cdots & \mathrm{Hom}_k\big(V_1, W_n\big) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Hom}_k\big(V_n, W_0\big) & \mathrm{Hom}_k\big(V_n, W_1\big) & \cdots & \mathrm{Hom}_k\big(V_n, W_n\big) \end{pmatrix} \tag{3.1.11}$$

For any $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$-morphism $(f^{\mathrm{op}} : V \to V', g : W \to W')$ and $L \in \hom\big(V, W\big)$

$$g \circ L \circ f \tag{3.1.12}$$

is an element of $\hom\big(V', W'\big)$. In other words

$$\hom(f^{\mathrm{op}}, g) : \hom\big(V, W\big) \longrightarrow \hom\big(V', W'\big) \ , \tag{3.1.13}$$

defined componentwise from that in Chapter 2 and is clearly k-linear and degree preserving:

$$\mathrm{hom}(f^{\mathrm{op}},g)\Big(\mathrm{hom}\big(V,W\big)_n\Big) \subset \bigoplus_{l\in\mathbb{Z}} \mathrm{hom}\big(V'_l,W'_{l+n}\big) = \mathrm{hom}\big(V',W'\big)_n \ . \tag{3.1.14}$$

The currying natural isomorphism $\zeta : \mathrm{Hom}_{\mathscr{M}}(-\otimes -,-) \Rightarrow \mathrm{Hom}_{\mathscr{M}}(-,\mathrm{hom}(-,-))$ of functors from $\mathscr{M}^{\mathrm{op}}\times\mathscr{M}^{\mathrm{op}}\times\mathscr{M} \to \mathsf{Sets}$ is defined componentwise from that in Chapter 2: Given any three objects V,W,X in $\mathscr{M}$ and $f\in\mathrm{Hom}_{\mathscr{M}}\big(V\otimes W,X\big)$ we have that $\zeta(f)$ is degree preserving since f is degree preserving:

$$\zeta(f)\big(V_n\big) = f\big(V_n\otimes -\big) \subset \bigoplus_{l\in\mathbb{Z}} \mathrm{Hom}_k\big(W_l,X_{n+l}\big) = \mathrm{hom}\big(W,X\big)_n \ . \tag{3.1.15}$$

Given $g\in\mathrm{Hom}_{\mathscr{M}}\big(V,\mathrm{hom}\big(W,X\big)\big)$, we have that $g\big(V_n\big)\subset\mathrm{hom}\big(W,X\big)_n$ since g is degree preserving and hence $\zeta^{-1}(g)$ is degree preserving:

$$\zeta^{-1}(g)(V_n\otimes W_m) = g\big(V_n\big)\big(W_m\big) \subset X_{m+n} \ . \tag{3.1.16}$$

Here and in the following we refrain from writing indices on the components of natural transformations.

Since the evaluation, internal composition and internal tensor product morphisms for internal hom-objects are defined by compositions of $\mathscr{M}$-morphisms they are degree preserving.

We note that due to the grading, the internal tensor product of internal homomorphisms evaluates on homogeneous elements as (cf. Equation (2.1.28))

$$\mathrm{ev}_{V\otimes X}(L \circledast P\otimes(-\otimes -) = (-1)^{|P|\,|\mathrm{comp}_V|}\mathrm{ev}(L\otimes(-))\otimes\mathrm{ev}(P\otimes(-)) \ , \tag{3.1.17}$$

where comp_V denotes the homogeneous component of V on which L may be evaluated, and that

$$(L\circledast P)\bullet(L'\circledast P') = (-1)^{|P||L'|}(L\bullet L'\circledast P\bullet P') \ , \tag{3.1.18}$$

(cf. Proposition 2.2.29). The degrees on either side match recalling that $|P \circ P'| = |P| + |P'|$.

Definition 3.1.2. Denote by $V[m]$ the object shifted in $\mathbb{Z}$-degree by m. That is

$$V[m] = \bigoplus_{n\in\mathbb{Z}} V_{n-m} \ . \tag{3.1.19}$$

3.1.2 Algebras and bimodules

The theory for algebras and bimodules in Chapter 2 extends exactly to the graded setting with the additional property that every instance of the braiding on homogenous elements of degree n and m gives rise to a sign

$$(-1)^{n\,m} \ . \tag{3.1.20}$$

Remark 3.1.3. We note that an algebra in $\mathscr{M}$ is a graded differential algebra (see e.g. [32]) since the multiplication is a morphism, i.e. it is degree preserving.

3.1.3 $\mathbb{Z}$-graded quasi-Hopf representation categories

In this chapter we shall view any quasitriangular quasi-Hopf algebra H as being $\mathbb{Z}$-graded and sitting in degree 0. A representation of H on an object V in $\mathscr{M}$ is an Alg-morphism

$$\rho_V : H \longrightarrow \mathrm{End}(V) \ . \tag{3.1.21}$$

The bounded $\mathbb{Z}$-graded representation category $[H, \mathscr{M}]$ of H is defined completely analogously to Chapter 2: The objects in $[H, \mathscr{M}]$ are functors ρ_V where now $\rho_V(*) = V$ is a bounded $\mathbb{Z}$-graded k-module. The morphisms in $[H, \mathscr{M}]$ are the H-equivariant $\mathscr{M}$-morphisms $f : V \to W$.

The closed braided monoidal structures discussed in Chapter 2 on $[H, \mathscr{M}]$ and on H–Bimod$(A)^{\mathrm{sym}}$, the category of symmetric bimodules over a commutative algebra

ρ_A in H–Alg$^{\mathrm{com}}$, extend exactly to the graded setting. We take note that the braiding on homogeneous elements of degree n and m in $[H, \mathscr{M}]$ now gives rise to a sign

$$(-1)^{n\,m} \tag{3.1.22}$$

recalling the definition of σ in Subsection 3.1.1.

The map-like structures for internal homomorphisms in $[H, \mathscr{M}]$ satisfy the properties discussed in Chapter 2. We note the following useful result derived from Lemma 2.2.28.

Lemma 3.1.4. *Let $\rho_V, \rho_W, \rho_X, \rho_Y, \rho_Z$ be any five objects in $[H, \mathscr{M}]$. Then for any $L \in \hom(V, W)$ and $K \in \hom(W, X)$ we have*

$$[K \circledast 1_Z, L \circledast 1_Z] = [K, L] \circledast 1_Z \ , \tag{3.1.23a}$$

$$[1_Z \circledast K, 1_Z \circledast L] = 1_Z \circledast [K, L] \ , \tag{3.1.23b}$$

$$[1_W \circledast K, L \circledast 1_W] = 0 \ , \tag{3.1.23c}$$

$$[L \circledast 1_X, 1_V \circledast K] = 0 \ . \tag{3.1.23d}$$

where $1_V := \rho_V(\beta) \in \hom(V, V)$, for all objects ρ_V in $[H, \mathscr{M}]$, are the unit internal homomorphisms.

Proof. The proof follows directly from Lemma 2.2.28 and also using the triangularity of the R-matrix for the last identity. □

Remark 3.1.5. To simplify notation, in what follows we shall drop the labels on the unit internal homomorphisms and simply write $1 := \rho_V(\beta)$, for any object ρ_V in $[H, \mathscr{M}]$.

3.2 Derivations and differential operators

In the remainder of this chapter we shall systematically build up notions of differential geometry internal to the bounded $\mathbb{Z}$-graded representation category $[H, \mathscr{M}]$ of a triangular quasi-Hopf algebra H. In this section we shall address the notions of derivations, differential operators and differential calculi. We describe derivations and differential operators as subobjects of the internal endomorphisms in $[H, \mathscr{M}]$ by expressing the algebraic properties which characterize them in terms of universal categorical constructions. See A.4 or [50] for the definition of limit and colimit in a category.

3.2.1 Derivations

We give a description of the derivations on an object ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$ by using universal constructions in the braided closed monoidal category $[H, \mathscr{M}]$ to formalize a suitable version of the Leibniz rule, that is compatible with the structures in $[H, \mathscr{M}]$, in terms of an equalizer.

In order to reformulate the notion of derivation on an algebra in the framework of the closed braided monoidal category $[H, \mathscr{M}]$, we note the following basic properties of derivations: (1) Derivations of a graded algebra belong to the collection of endomorphisms of the algebra. (2) Derivations of a graded algebra obey a graded Leibniz rule.

We also have the following requirement when there is an action of a quasi-Hopf algebra H on the algebra: we do not wish the derivations to preserve the H-module structure of the algebra ρ_A but rather to be able to be transformed under it i.e. the derivations ought to be an H-module and hence condition (1) above should be refined to $\mathrm{der}(\rho_A) \subset \hom(\rho_A, \rho_A) =: \mathrm{end}(\rho_A)$.

Now we notice that the graded Leibniz rule in $\mathscr{M}$ can be written as an operator equation (viewing D now as it should be as an internal homomorphism in $\mathscr{M}$ and recalling that the commutator contains a sign from the braiding):

$$[D, x](y) := [D, \widehat{l}_A(x)](y) = \widehat{l}_A(\mathrm{ev}(D \otimes x))(y) \ , \tag{3.2.1}$$

with $l_A : A \otimes A \to A$ the left A-action induced by the product in A. We recall that for any object ρ_V in H–Bimod$(A)^{\mathrm{sym}}$ the $[H,\mathscr{M}]$-morphism

$$\widehat{l}_V := \zeta(l_V) : \rho_A \Longrightarrow \mathrm{end}(\rho_V)\ , \tag{3.2.2}$$

which is obtained by currying the left ρ_A-action $l_V : \rho_A \otimes \rho_V \Rightarrow \rho_V$ is an H–Alg-morphism to the algebra of internal endomorphisms, cf. Example 2.2.20.

The graded Leibniz rule can therefore be captured by the following equality of maps

$$[\,\cdot\,,\,\cdot\,] \circ (\mathrm{id} \otimes \widehat{l}_A) = \widehat{l}_A \circ \mathrm{ev} : \mathrm{end}(\rho_A) \otimes \rho_A \Longrightarrow \mathrm{end}(\rho_A)\ . \tag{3.2.3}$$

The maps $[\,\cdot\,,\,\cdot\,], \widehat{l}_A$ and ev need to preserve the H-module structure, i.e. be morphisms if their target objects are to be H-modules as we require. So the Leibniz rule is captured by equating two morphisms in $[H,\mathscr{M}]$. We organise this information as follows: The space of derivations is a subset of the internal endomorphisms on ρ_A for which there are two equal (parallel) morphisms

$$\begin{tikzcd} \mathrm{end}(\rho_A) \otimes \rho_A \arrow[r, Rightarrow, shift left, "{[\cdot\,,\cdot]}"] \arrow[r, Rightarrow, shift right, "\widehat{l}_A \circ \mathrm{ev}"'] & \mathrm{end}(\rho_A) \end{tikzcd} \tag{3.2.4}$$

where for brevity we denote by $[\,\cdot\,,\,\cdot\,]$ the composition $[\,\cdot\,,\,\cdot\,] \circ \big(\mathrm{id} \otimes \widehat{l}_A\,\big)$. We use the currying map for internal homomorphisms to rewrite this as

$$\begin{tikzcd} \mathrm{end}(\rho_A) \arrow[r, Rightarrow, shift left, "{\zeta([\cdot,\cdot])}"] \arrow[r, Rightarrow, shift right, "\zeta(\widehat{l}_A \circ \mathrm{ev})"'] & \mathrm{hom}(\rho_A, \mathrm{end}(\rho_A)) \end{tikzcd} \tag{3.2.5}$$

We have the following

Definition 3.2.1 (Derivations)**.** Let ρ_A be an object in H–Alg$^{\mathrm{com}}$. The *derivations of* ρ_A is the object $\mathrm{der}(\rho_A)$ in $[H,\mathscr{M}]$ which is defined by the equalizer

$$\begin{tikzcd} \mathrm{der}(\rho_A) \arrow[r, Rightarrow, "\iota"] & \mathrm{end}(\rho_A) \arrow[r, Rightarrow, shift left, "{\zeta([\cdot,\cdot])}"] \arrow[r, Rightarrow, shift right, "\zeta(\widehat{l}_A \circ \mathrm{ev})"'] & \mathrm{hom}(\rho_A, \mathrm{end}(\rho_A)) \end{tikzcd} \tag{3.2.6}$$

in $[H,\mathscr{M}]$ where ι is the inclusion morphism.

In the category $[H, \mathscr{M}]$ equalizers may be computed by taking the kernel of the difference of the two parallel morphisms. In particular, $\mathrm{der}(\rho_A)$ can be represented explicitly as the kernel

$$\mathrm{der}(\rho_A) = \mathrm{Ker}\Big(\zeta\big([\,\cdot\,,\,\cdot\,] - \widehat{l} \circ \mathrm{ev}\big)\Big) \ . \tag{3.2.7}$$

The following lemma allows us to establish a relation between our definition of derivations and the standard definition in terms of a Leibniz rule.

Lemma 3.2.2. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$. An $[H, \mathscr{M}]$-subobject $\rho_U \subseteq \mathrm{end}(\rho_A)$ is an $[H, \mathscr{M}]$-subobject of $\mathrm{der}(\rho_A)$ if and only if*

$$[L, a] = \widehat{l}(\mathrm{ev}(L \otimes a)) \ , \tag{3.2.8}$$

for all $L \in U$ and $a \in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,] - \widehat{l} \circ \mathrm{ev} : \mathrm{end}(\rho_A) \otimes \rho_A \Rightarrow \mathrm{end}(\rho_A)$ and $j : \rho_U \Rightarrow \mathrm{end}(\rho_A)$ the inclusion $[H, \mathscr{M}]$-morphism, we have to show that $\zeta(f) \circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}) = 0$. This is a consequence of item (ii) of Lemma 2.1.26. □

In summary we have (1) $\mathrm{der}(\rho_A)$ is an $[H, \mathscr{M}]$-object and (2) elements of $\mathrm{der}(\rho_A)$ satisfy a suitable generalization of the graded Leibniz rule that is consistent with the structures in the braided closed monoidal category $[H, \mathscr{M}]$. That is we have correctly formulated the notion of derivation in the framework of the closed braided monoidal category $[H, \mathscr{M}]$.

Finally, we prove a structural result for derivations.

Proposition 3.2.3. *Let H be a triangular quasi-Hopf algebra and ρ_A any object in H–$\mathrm{Alg}^{\mathrm{com}}$. Then the $[H, \mathscr{M}]$-object given by the derivations $\mathrm{der}(\rho_A)$, together with the internal commutator $[\,\cdot\,,\,\cdot\,]$ given in (2.1.39) interpreted in the closed braided monoidal category $[H, \mathscr{M}]$ in Chapter 2, is a Lie algebra in $[H, \mathscr{M}]$.*

Proof. We already know from Corollary 2.1.14 in Chapter 2 that, under our hypotheses, $\mathrm{end}(\rho_A)$ together with the internal commutator $[\,\cdot\,,\,\cdot\,]$ is a Lie algebra in

$[H,\mathscr{M}]$. Moreover, $\mathrm{der}(\rho_A)$ is by construction an $[H,\mathscr{M}]$-subobject of $\mathrm{end}(\rho_A)$, so it remains to prove that the image of the restricted internal commutator

$$[\,\cdot\,,\,\cdot\,] : \mathrm{der}(\rho_A) \otimes \mathrm{der}(\rho_A) \Longrightarrow \mathrm{end}(\rho_A) \tag{3.2.9}$$

is an $[H,\mathscr{M}]$-subobject of $\mathrm{der}(\rho_A)$. Using Lemma 3.2.2 this is the case if and only if

$$[\,\cdot\,,\,\cdot\,] \circ ([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}_A) = \widehat{l} \circ \mathrm{ev} \circ ([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id})\ , \tag{3.2.10}$$

One can easily show that this equality holds true by using the braided Jacobi identity and antisymmetry (cf. items (ii) and (i) of Proposition 2.1.13) and the derivation property of Lemma 3.2.2:

$$\begin{aligned}
[\,\cdot\,,\,\cdot\,] \circ ([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}_A) &= -[\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \big((\tau \circ \Phi) + (\Phi^{-1} \circ \tau)\big) \\
&= [\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \big(\Phi + (\Phi^{-1} \circ \tau_{1,3} \circ \Phi)\big) \\
&= [\,\cdot\,,\,\cdot\,] \circ \big(\widehat{l} \circ \mathrm{ev} \otimes \mathrm{id}\big) \circ \big(\Phi + (\Phi^{-1} \circ \tau_{1,3} \circ \Phi)\big) \\
&= [\,\cdot\,,\,\cdot\,] \circ \big(\mathrm{id} \otimes \widehat{l} \circ \mathrm{ev}\big) \circ \big(\Phi - (\tau_{1,23} \circ \Phi \circ \tau_{1,2})\big) \\
&= \widehat{l} \circ \mathrm{ev} \circ \big(\mathrm{id} \otimes \mathrm{ev}\big) \circ \big(\Phi - (\Phi \circ \tau_{1,2})\big) \\
&= \widehat{l} \circ \mathrm{ev} \circ \big(\mathrm{id} \otimes \mathrm{ev}\big) \circ \Phi \circ \big(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} - \tau_{1,2}\big) \\
&= \widehat{l} \circ \mathrm{ev} \circ \big(\bullet \otimes \mathrm{id}\big) \circ \big(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} - \tau_{1,2}\big) \\
&= \widehat{l} \circ \mathrm{ev} \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big)\ .
\end{aligned} \tag{3.2.11}$$

The first equality follows from the braided Jacobi identity item (ii) of Proposition 2.1.13, the second equality follows from the braided antisymmetry item (i) of Proposition 2.1.13 and the R-matrix properties (2.1.103b), (2.1.103c), the third equality follows from Lemma 3.2.2, the fourth equality follows from inserting $\mathrm{id} = \tau^{-1} \circ \tau$ and using the braided antisymmetry item (i) of Proposition 2.1.13, the fifth equality follows from Lemma 3.2.2, the R-matrix properties (2.1.103c) together with triangularity of the R-matrix ($\tau^2 = \mathrm{id}$), the seventh equality follows from item (ii) of Proposition 2.2.13. (Subscripts on the braiding indicate which components of the tensor products are being braided.) □

3.2.2 Cochain twisting of derivations

We shall briefly study the deformation of derivations under cochain twisting.

Let H be a quasitriangular quasi-Hopf algebra and F a cochain twisting element. In Chapter 2 we saw that the braided closed monoidal functor $\mathcal{F} : [H,\mathscr{M}] \to [H_F,\mathscr{M}]$ induces functors (denoted with abuse of notation by the same symbols) $\mathcal{F} : H\text{–Alg}^{\mathrm{com}} \to H_F\text{–Alg}^{\mathrm{com}}$ and $\mathcal{F} : H\text{–Bimod}(A)^{\mathrm{sym}} \to H_F\text{–Bimod}(A_F)^{\mathrm{sym}}$, which allow us to twist quantize algebras and bimodules in $[H,\mathscr{M}]$ to algebras and bimodules in $[H_F,\mathscr{M}]$.

Proposition 3.2.4. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$ and F any cochain twisting element based on H. Then the coherence map $\gamma : \mathrm{end}_F(\mathcal{F}(\rho_A)) \Rightarrow \mathcal{F}(\mathrm{end}(\rho_A))$ restricts to an $[H_F,\mathscr{M}]$-isomorphism*

$$\gamma : \mathrm{der}_F(\mathcal{F}(\rho_A)) \Longrightarrow \mathcal{F}(\mathrm{der}(\rho_A)) \ . \tag{3.2.12}$$

Proof. The proof follows that of Proposition 2.3.15 in Chapter 2 and it requires showing commutativity of the ${}^{H_F}\mathscr{M}$-diagram

$$\begin{tikzcd}
\mathrm{end}_F(\mathcal{F}(\rho_A)) \arrow[r, Rightarrow, shift left, "\zeta_F([\,\cdot\,,\,\cdot\,]_F)"] \arrow[r, Rightarrow, shift right, "\zeta_F(\widehat{l}_F\circ\mathrm{ev}_F)"'] \arrow[dd, Rightarrow, "\gamma"'] & \mathrm{hom}_F(\mathcal{F}(\rho_A), \mathrm{end}_F(\mathcal{F}(\rho_A))) \arrow[d, Rightarrow, "\gamma\circ(\,\cdot\,)"] \\
 & \mathrm{hom}_F(\mathcal{F}(\rho_A), \mathcal{F}(\mathrm{end}(\rho_A))) \arrow[d, Rightarrow, "\gamma"] \\
\mathcal{F}(\mathrm{end}(\rho_A)) \arrow[r, Rightarrow, shift left, "\mathcal{F}(\zeta([\,\cdot\,,\,\cdot\,]))"] \arrow[r, Rightarrow, shift right, "\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}))"'] & \mathcal{F}(\mathrm{hom}(\rho_A, \mathrm{end}(\rho_A)))
\end{tikzcd} \tag{3.2.13}$$

Showing the commutativity of diagram (3.2.13) entails showing that

$$\gamma \circ \zeta_F([\cdot,\cdot]_F) = \mathcal{F}(\zeta([\cdot,\cdot])) \ , \tag{3.2.14a}$$

$$\gamma \circ \zeta_F(\widehat{l}_{A_F} \circ \mathrm{ev}_F) = \mathcal{F}(\zeta(\widehat{l}_A \circ \mathrm{ev})) \ . \tag{3.2.14b}$$

(Note that due to the H_F-equivariance of $\zeta_F([\cdot,\cdot]_F)$ and $\zeta_F(\widehat{l}_{A_F} \circ \mathrm{ev}_F)$ two instances of γ in the diagram (3.2.13) cancel.) Note that (3.2.14a) was shown in (2.3.60) in the section on cochain twisting of internal hom-objects in H–Bimod$(A)^{\mathrm{sym}}$ in Chapter 2. To prove equation (3.2.14b) we observe that

$$
\begin{aligned}
\gamma \circ \zeta_F(\widehat{l}_{A_F} \circ \mathrm{ev}_F) &= \gamma \circ \widehat{l}_{A_F} \circ \mathrm{ev}_F \circ \zeta_F(\mathrm{id}) \\
&= \gamma \circ \gamma^{-1} \circ \mathcal{F}(\widehat{l}_A) \circ \mathcal{F}(\mathrm{ev}) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \circ \varphi^{-1} \circ (\gamma^{-1} \otimes_F \mathrm{id}) \circ \mathcal{F}(\zeta(\mathrm{id})) \\
&= \mathcal{F}(\widehat{l}_A \circ \mathrm{ev} \circ \zeta(\mathrm{id})) \\
&= \mathcal{F}(\zeta(\widehat{l}_A \circ \mathrm{ev})) \ .
\end{aligned} \tag{3.2.15}
$$

The first and last equalities follows from (2.2.88), the second equality follows from (2.3.54a), (2.2.104a) and (2.2.101), and the third equality follows from the functoriality of $\mathcal{F}$. □

In summary, the twist deformation quantisation functor preserves derivations. That is there is a structural isomorphism between derivations in the closed braided monoidal category $[H, \mathscr{M}]$ and the closed braided monoidal category $[H_F, \mathscr{M}]$, and the H-actions required to construct this isomorphism are precisely those which preserve the internal endomorphism objects.

3.2.3 Differential operators and calculi

We provide in this section a description of differential calculus in $[H, \mathscr{M}]$ as a first step towards a description of connection in $[H, \mathscr{M}]$. We already have the notion of exterior graded algebra (this is simply a commutative algebra object in $[H, \mathscr{M}]$, cf. Remark 3.1.3). In order to provide a categorical description of exterior derivative we require the notion of differential operator in $[H, \mathscr{M}]$. As a basis for a categorical formulation we use the abstract definition of differential operators provided by Grothendieck: See [44, pp. 15 - 18] for a detailed discussion. Although this is a very abstract definition, it captures the essential properties of a differential operator and is equivalent to the usual definition in a local coordinate basis. Since coordinate transformations preserve the order of differential operators, it is possible to give a basis-independent definition of differential operators of different orders.

Let ρ_A be an object in H–Alg$^{\mathrm{com}}$ and ρ_V any object in H–Bimod$(A)^{\mathrm{sym}}$. We define the internal multi-commutator of order $n \in \mathbb{Z}_{>0}$ to be the $[H,\mathscr{M}]$-morphism

$$[\,\cdot\,,\,\cdot\,]^{(n)} : \big(\cdots\big((\mathrm{end}(\rho_V)\otimes\rho_A)\otimes\rho_A\big)\cdots\big)\otimes\rho_A \Longrightarrow \mathrm{end}(\rho_V)\ , \tag{3.2.16a}$$

where the source contains n factors of ρ_A, given by the composition

$$[\,\cdot\,,\,\cdot\,]^{(n)} := [\,\cdot\,,\,\cdot\,]\circ\big([\,\cdot\,,\,\cdot\,]\otimes\mathrm{id}\big)\circ\cdots\circ\big((\cdots(([\,\cdot\,,\,\cdot\,]\otimes\mathrm{id})\otimes\mathrm{id})\cdots)\otimes\mathrm{id}\big)\ . \tag{3.2.16b}$$

We have suppressed as before the precomposition of the internal multi-commutator with $(\cdots((\mathrm{id}\otimes\widehat{l}\,)\otimes\widehat{l}\,)\cdots)\otimes\widehat{l}$, where $\widehat{l}$ is the H–Alg-morphism given in (3.2.2). We further denote by $\Phi^{(-n)}$ the combination of associators required to re-bracket the expressions

$$\begin{array}{c}\mathrm{end}(\rho_V)\otimes\big(\rho_A\otimes(\rho_A\otimes(\cdots(\rho_A\otimes\rho_A)\cdots))\big)\\ \Big\Downarrow{\scriptstyle \Phi^{(-n)}}\\ \big(\cdots\big((\mathrm{end}(\rho_V)\otimes\rho_A)\otimes\rho_A\big)\cdots\big)\otimes\rho_A\end{array} \tag{3.2.17}$$

where again the source and target contain n factors of ρ_A. We shall denote the source of this $[H,\mathscr{M}]$-isomorphism by $\mathrm{end}(\rho_V)\otimes\rho_A^{\otimes n}$.

Definition 3.2.5 (Differential operators)**.** Let ρ_A be an object in H–Alg$^{\mathrm{com}}$ and ρ_V any object in H–Bimod$(A)^{\mathrm{sym}}$. The *differential operators of order $n \in \mathbb{Z}_{\geq 0}$ of ρ_V* is the object $\mathrm{diff}^n(\rho_V)$ in $[H,\mathscr{M}]$ which is defined by the equalizer

$$\mathrm{diff}^n(\rho_V) \Longrightarrow \mathrm{end}(\rho_V) \underset{0}{\overset{\zeta([\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\Phi^{(-(n+1))})}{\rightrightarrows}} \mathrm{hom}(\rho_A^{\otimes n},\mathrm{end}(\rho_V)) \tag{3.2.18}$$

in $[H,\mathscr{M}]$. This equalizer can be realized explicitly in terms of the $[H,\mathscr{M}]$-subobject

$$\mathrm{diff}^n(\rho_V) = \mathrm{Ker}\big(\zeta\big([\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\Phi^{(-(n+1))}\big)\big) \tag{3.2.19}$$

of the internal endomorphism object $\mathrm{end}(\rho_V)$ in $[H,\mathscr{M}]$.

Remark 3.2.6. Comparing the Definitions 3.2.5 and 2.3.10 we observe that the order 0 differential operators $\mathrm{diff}^0(\rho_V)$ are the internal endomorphisms $\mathrm{end}_A(\rho_V)$ in the category H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Lemma 3.2.7. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$ and let ρ_V be any object in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. An $[H,\mathscr{M}]$-subobject $\rho_U \subseteq \mathrm{end}(\rho_V)$ is an $[H,\mathscr{M}]$-subobject of $\mathrm{diff}^n(\rho_V)$ if and only if*

$$\big[\big[\cdots\big[[L,a_1],a_2\big],\cdots\big],a_{n+1}\big] = 0 \ , \tag{3.2.20}$$

for all $L \in U$ and $a_1,a_2,\dots,a_{n+1} \in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,]^{(n+1)} \circ \Phi^{(-(n+1))} : \mathrm{end}(\rho_V) \otimes \rho_A^{\otimes n} \Rightarrow \mathrm{end}(\rho_V)$ and $j : \rho_U \Rightarrow \mathrm{end}(\rho_V)$ the inclusion $[H,\mathscr{M}]$-morphism, it follows from Lemma 2.1.26 (ii) that $\zeta(f)\circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}) = 0$. The latter condition is equivalent to $[\,\cdot\,,\,\cdot\,]^{(n+1)} \circ \big((\cdots((j \otimes \mathrm{id}) \otimes \mathrm{id})\cdots) \otimes \mathrm{id}\big) \circ \Phi^{(-(n+1))} = 0$, and the assertion now follows because $\Phi^{(-(n+1))}$ is an isomorphism. $\square$

There is an $[H,\mathscr{M}]$-subobject relation $\mathrm{diff}^n(\rho_V) \subseteq \mathrm{diff}^m(\rho_V)$ for all $n \leq m$, which immediately follows from Lemma 3.2.7 and (3.2.19). These subobject relations give rise to the sequence of $[H,\mathscr{M}]$-monomorphisms

$$\mathrm{diff}^0(\rho_V) \Longrightarrow \mathrm{diff}^1(\rho_V) \Longrightarrow \mathrm{diff}^2(\rho_V) \Longrightarrow \ \cdots \ \Longrightarrow \mathrm{diff}^n(\rho_V) \Longrightarrow \ \cdots \ . \tag{3.2.21}$$

We shall now show that differential operators can be composed with respect to the internal composition.

Proposition 3.2.8. *The internal composition $\bullet : \mathrm{end}(\rho_V) \otimes \mathrm{end}(\rho_V) \Rightarrow \mathrm{end}(\rho_V)$ restricts to an $[H,\mathscr{M}]$-morphism*

$$\bullet : \mathrm{diff}^n(\rho_V) \otimes \mathrm{diff}^m(\rho_V) \Longrightarrow \mathrm{diff}^{n+m}(\rho_V) \ , \tag{3.2.22}$$

for all $n,m \in \mathbb{Z}_{\geq 0}$.

Proof. Restricting $\bullet : \mathrm{end}(\rho_V) \otimes \mathrm{end}(\rho_V) \Rightarrow \mathrm{end}(\rho_V)$ to the corresponding $[H, \mathscr{M}]$-subobjects of differential operators yields an $[H, \mathscr{M}]$-morphism $\bullet : \mathrm{diff}^n(\rho_V) \otimes \mathrm{diff}^m(\rho_V) \Rightarrow \mathrm{end}(\rho_V)$ and we have to prove that its image lies in $\mathrm{diff}^{n+m}(\rho_V)$. As the image of this $[H, \mathscr{M}]$-morphism is an $[H, \mathscr{M}]$-subobject of $\mathrm{end}(\rho_V)$, by Lemma 3.2.7 it is enough to show that

$$\big[\big[\cdots\big[[L \bullet L', a_1], a_2\big], \cdots\big], a_{n+m+1}\big] = 0 \ , \tag{3.2.23}$$

for all $L \in \mathrm{diff}^n(V)$, $L' \in \mathrm{diff}^m(V)$ and $a_1, a_2, \ldots, a_{n+m+1} \in A$. This equality follows by iteratively using the derivation property of the internal commutator, cf. item (iii) of Proposition 2.1.13, and applying Lemma 3.2.7 to L and L'. See B.5 for further details. □

Forming the colimit in $[H, \mathscr{M}]$ of the diagram given in (3.2.21) we can define the object $\mathrm{diff}(\rho_V)$ of differential operators on ρ_V. This colimit can be represented explicitly as the union of differential operators of all orders $n \in \mathbb{Z}_{\geq 0}$, i.e.

$$\mathrm{diff}(\rho_V) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathrm{diff}^n(\rho_V) \subseteq \mathrm{end}(\rho_V) \ . \tag{3.2.24}$$

Corollary 3.2.9. *The differential operators* $\mathrm{diff}(\rho_V)$ *is an* H–Alg*-subobject of the algebra of internal endomorphisms* $\mathrm{end}(\rho_V)$ *(cf. Example 2.2.20).*

Proof. By Proposition 3.2.8 the internal composition closes on $\mathrm{diff}(\rho_V)$, i.e. there is an $[H, \mathscr{M}]$-morphism

$$\bullet : \mathrm{diff}(\rho_V) \otimes \mathrm{diff}(\rho_V) \Longrightarrow \mathrm{diff}(\rho_V) \ . \tag{3.2.25}$$

The unit $\eta : \rho_I \Rightarrow \mathrm{end}(\rho_V)$ has its image in the degree 0 differential operators because of the calculation

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]\circ(1\otimes\mathrm{id}_A) &= \vartheta(\mathrm{id})\bullet\widehat{l}(-)-\widehat{l}(-)\bullet\vartheta(\mathrm{id})\\ &= \widehat{l}(-)-\widehat{l}(-)\\ &= 0\ ,\end{aligned} \tag{3.2.26}$$

recalling that $1=\rho_V(\beta)=\vartheta(\mathrm{id})$ from 2.2.15, and Lemma 3.2.7; here we used the normalization $(\epsilon\otimes\mathrm{id})(R)=1$ of the R-matrix and the property in item (iii) of Proposition 2.2.38. □

Remark 3.2.10. Combining Lemmas 3.2.2 and 3.2.7 we see that for any object ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$, $\mathrm{der}(\rho_A)\subseteq\mathrm{diff}^1(\rho_A)$ is an $[H,\mathscr{M}]$-subobject, i.e. the derivations of ρ_A are differential operators of order 1.

With the techniques developed above we can now introduce the notion of exterior derivative for a graded differential algebra in $[H,\mathscr{M}]$. To begin with, we require to understand the type of object in $[H,\mathscr{M}]$ the correct generalisation of exterior derivative is: The classical exterior derivative commutes with the Lie derivative and since in applications any H-action is implemented by Lie derivatives, the exterior derivative commutes with H-actions; in this formalism, it is H-equivariant. However the exterior derivative cannot be a morphism in the graded category $[H,\mathscr{M}]$ since it is by definition a linear map of degree 1. We have already seen in Subsection 2.2.15 that H-invariant internal homomorphisms (of degree 0) can be identified with morphisms and the same constructions show that H-invariant internal homomorphism of any degree can be identified with H-equivariant maps (of the corresponding degree). Hence the correct categorical structure with which to describe an exterior derivative in $[H,\mathscr{M}]$ is an H-invariant internal homomorphism. In order to fit later into the definition of connection in terms of an equaliser of morphisms in $[H,\mathscr{M}]$, we define below a morphism whose target contains the exterior derivative in $[H,\mathscr{M}]$.

Recalling Definition 3.1.2 we denote by $I[1]$ the object in $\mathscr{M}$ which is obtained by shifting the unit object $I=k$ in $\mathbb{Z}$-degree by 1: $I[1]_1=k$ and $I[1]_n=0$, for all $n\neq 1$.

Definition 3.2.11 (Differential calculus)**.** Let H be a quasitriangular quasi-Hopf algebra. A *differential calculus* (ρ_A, d) in $[H, \mathscr{M}]$ is an object ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$ together with an $[H, \mathscr{M}]$-morphism $\mathrm{d} : \rho_{I[1]} \to \mathrm{der}(\rho_A)$ which is nilpotent in the sense that the composition of $[H, \mathscr{M}]$-morphisms

$$\rho_{I[1]} \otimes \rho_{I[1]} \xrightarrow{\mathrm{d}\otimes\mathrm{d}} \mathrm{der}(\rho_A) \otimes \mathrm{der}(\rho_A) \Longrightarrow \mathrm{diff}(\rho_A) \otimes \mathrm{diff}(\rho_A) \xRightarrow{\bullet} \mathrm{diff}(\rho_A) \quad (3.2.27)$$

is 0; here the second arrow is defined using Remark 3.2.10.

Remark 3.2.12. Given a differential calculus (ρ_A, d) in $[H, \mathscr{M}]$ there is a distinguished H-invariant derivation of $\mathbb{Z}$-degree 1, which is given by $\mathrm{d}(1) \in \mathrm{der}(\rho_A)$ and is called the differential.

3.2.4 Cochain twisting of differential operators and calculi

The cochain twist deformation quantization functor preserves differential operators and differential calculi.

Proposition 3.2.13. *Let ρ_A be any object in H–$\mathrm{Alg}^{\mathrm{com}}$, let ρ_V be any object in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ and let F be any cochain twisting element based on H. Then the coherence map $\gamma : \mathrm{end}_F(\mathcal{F}(\rho_V)) \Rightarrow \mathcal{F}(\mathrm{end}(\rho_V))$ restricts to an $[H_F, \mathscr{M}]$-isomorphism*

$$\gamma : \mathrm{diff}^n_F(\mathcal{F}(\rho_V)) \longrightarrow \mathcal{F}(\mathrm{diff}^n(\rho_V)) \ , \quad (3.2.28)$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 3.2.4. One is required to show that

$$\gamma \circ \zeta_F\big([\,\cdot\,,\,\cdot\,]^{(n+1)} \circ \Phi_F^{-(n+1)}\big) \circ \varphi^{(n+1)} = \mathcal{F}\big(\zeta\big([\,\cdot\,,\,\cdot\,]^{(n+1)} \circ \Phi_F^{-(n+1)}\big)\big) \ . \quad (3.2.29)$$

We recall equation (2.3.54b) in which the $[H_F, \mathscr{M}]$-morphisms $[\cdot,\cdot]_F$ and $\mathcal{F}([\cdot,\cdot])$ are shown to be isomorphic to each other according to

$$[\cdot,\cdot]_F = \gamma^{-1} \circ \mathcal{F}\big([\cdot,\cdot]\big) \circ \varphi \circ (\gamma \otimes_F \mathrm{id}) \ . \quad (3.2.30)$$

It follows that the n-th order commutators $[\cdot,\cdot]_F^{(n)}$ and $\mathcal{F}\big([\cdot,\cdot]^{(n)}\big)$ can be written in terms of each other by iteratively applying the formula (3.2.30). We use the fact that the commutator acts on H-modules (which close under the action of H) to complete the proof. □

Proposition 3.2.14. *Let* $\big(\rho_A, \mathrm{d} : \rho_{I[1]} \to \mathrm{der}(\rho_A)\big)$ *be a differential calculus in* $[H, \mathscr{M}]$ *and let* F *be a cochain twisting element based on* H*. Then* $\mathcal{F}(\rho_A)$ *together with the* $[H_F, \mathscr{M}]$*-morphism*

$$\mathrm{d}_F := \gamma^{-1} \circ \mathcal{F}(\mathrm{d}) \circ \psi : \rho_{I_F[1]} \Longrightarrow \mathrm{der}_F(\mathcal{F}(\rho_A)) \tag{3.2.31}$$

is a differential calculus in $[H_F, \mathscr{M}]$*, where* ψ *is the coherence morphism in (2.2.24b).*

Proof. By Proposition 3.2.4, the target of d_F is as claimed in (3.2.31). Moreover, d_F is nilpotent (in $\mathrm{diff}_F(\mathcal{F}(\rho_A))$) because of the short calculation

$$\begin{aligned}
&\bullet_F \circ(\mathrm{d}_F(-) \otimes_F \mathrm{d}_F(-)) \\
&= \gamma^{-1} \circ \mathcal{F}(\bullet) \circ \varphi \circ (\gamma \otimes_F \gamma) \circ (\gamma^{-1} \otimes_F \gamma^{-1}) \circ (\mathcal{F}(\mathrm{d})(-) \otimes_F \mathcal{F}(\mathrm{d})(-)) \\
&= \gamma^{-1} \circ \mathcal{F}\big(\bullet \circ(\mathrm{d}(-) \otimes \mathrm{d}(-)\big) \\
&= 0 \ ,
\end{aligned} \tag{3.2.32}$$

In the first equality we have used (2.2.104b) and the definition of d_F (3.2.31) (noting that ψ is the identity), in the second equality we have used the definition of the coherence map (2.2.24a) and in the last equality the nilpotency of d. □

3.3 Connections

For a given differential calculus (ρ_A, d) in $[H, \mathscr{M}]$, we shall develop the notion of connections on objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ by again using universal constructions in the category $[H, \mathscr{M}]$. We show that connections of objects ρ_V, ρ_W in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ can be canonically lifted to connections on the tensor product object $\rho_V \otimes_A \rho_W$ and on the internal hom-object $\mathrm{hom}_A(\rho_V, \rho_W)$.

3.3.1 Connections on symmetric bimodules

In order to place the notion of connection in the framework of the closed braided monoidal category $[H, \mathscr{M}]$, we recall that (1) connections are linear maps $\nabla : V \to V \otimes \Omega^1$ of modules over a commutative algebra Ω^0 where $\Omega = \bigoplus_n \Omega^n$ is a graded differential algebra which (2) satisfy a graded Leibniz rule with respect to an exterior differential d for Ω: $\nabla(a\,v) = a\,\nabla(v) + v \otimes_A \mathrm{d}\,a$ for all $v \in V, a \in \Omega^0$. In our framework, graded differential algebras are commutative algebra objects ρ_A in $[H, \mathscr{M}]$ (cf. Remark 3.1.3) and the tensor product $\otimes_A$ for ρ_A-bimodules objects in $[H, \mathscr{M}]$ comes equipped with a right unitor which enables one to identify $\rho_V \otimes_A \rho_A \cong \rho_V$. Therefore in $[H, \mathscr{M}]$ connections are (1) endomorphisms on a symmetric bimodule object ρ_V in H–Bimod$(A)^{\mathrm{sym}}$ over a commutative algebra object ρ_A in H–Alg$^{\mathrm{com}}$ which (2) satisfy a suitable generalisation of the Leibniz rule compatible with the structures in $[H, \mathscr{M}]$ with respect to the exterior differential d(1) in Definition 3.2.11 (cf. equation (3.2.1))

$$[\nabla\,,\,a] := [\nabla, \widehat{l}_V(a)] = \widehat{l}_V(\mathrm{ev}(\mathrm{d}(1) \otimes a))\ , \tag{3.3.1}$$

for $a \in A$. Upon analysing this equation we notice that d(1) is not a morphism, but if we allow for nilpotent derivations $\mathrm{d}(c)$ for arbitrary $c \in k$, then (3.3.1) can be written as an equation of $[H, \mathscr{M}]$-morphisms:

$$[\,\cdot\,,\,\cdot\,] : \mathrm{end}(\rho_V) \otimes \rho_A \Longrightarrow \mathrm{end}(\rho_V)\ , \tag{3.3.2}$$

(given by the bracket (2.3.34)) from the left hand side, and

$$\widehat{l}_V \circ \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}_{\rho_A}) : \rho_{I[1]} \otimes \rho_A \Longrightarrow \mathrm{end}(\rho_V)\ , \tag{3.3.3}$$

from the right hand side. We write this equality with a unified source object

$$(\mathrm{end}(\rho_V) \times \rho_{I[1]}) \otimes \rho_A \tag{3.3.4}$$

using the projection $[H,\mathscr{M}]$-morphisms

$$\mathrm{pr}_1 : \mathrm{end}(\rho_V) \times \rho_{I[1]} \Longrightarrow \mathrm{end}(\rho_V) ~, \quad \mathrm{pr}_2 : \mathrm{end}(\rho_V) \times \rho_{I[1]} \Longrightarrow \rho_{I[1]} ~. \tag{3.3.5}$$

(These are morphisms since we define the H-action componentwise on the categorical product $\times$ in $[H,\mathscr{M}]$.) There are two parallel $[H,\mathscr{M}]$-morphisms

$$\big(\mathrm{end}(\rho_V) \times \rho_{I[1]}\big) \otimes \rho_A \underset{\widehat{l}\circ \mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id})}{\overset{[\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id})}{\rightrightarrows}} \mathrm{end}(\rho_V) ~. \tag{3.3.6}$$

Definition 3.3.1 (Connections)**.** Let $\big(\rho_A, \mathrm{d} : \rho_{I[1]} \Rightarrow \mathrm{der}(\rho_A)\big)$ be a differential calculus in $[H,\mathscr{M}]$ and ρ_V any object in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. The *connections of* ρ_V is the object $\mathrm{con}(\rho_V)$ in $[H,\mathscr{M}]$ which is defined by the equalizer

$$\mathrm{con}(\rho_V) \Longrightarrow \mathrm{end}(\rho_V) \times \rho_{I[1]} \underset{\zeta(\widehat{l}\circ \mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id}))}{\overset{\zeta([\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id}))}{\rightrightarrows}} \mathrm{hom}(\rho_A, \mathrm{end}(\rho_V)) \tag{3.3.7}$$

in $[H,\mathscr{M}]$. This equalizer can be realized explicitly in terms of the $[H,\mathscr{M}]$-subobject

$$\mathrm{con}(\rho_V) = \mathrm{Ker}\Big(\zeta\big([\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \otimes \mathrm{id}) - \widehat{l} \circ \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}) \circ (\mathrm{pr}_2 \otimes \mathrm{id})\big)\Big) \tag{3.3.8}$$

of the object $\mathrm{end}(\rho_V) \times \rho_{I[1]}$ in $[H,\mathscr{M}]$.

Lemma 3.3.2. *Let* (ρ_A, d) *be a differential calculus in* $[H,\mathscr{M}]$ *and* ρ_V *an object in* H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$*. An* $[H,\mathscr{M}]$*-subobject* $\rho_U \subseteq \mathrm{end}(\rho_V) \times \rho_{I[1]}$ *is an* $[H,\mathscr{M}]$*-subobject of* $\mathrm{con}(\rho_V)$ *if and only if*

$$[L, a] = \widehat{l}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) ~, \tag{3.3.9}$$

for all $(L, c) \in U$ *and* $a \in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \otimes \mathrm{id}) - \widehat{l} \circ \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}) \circ (\mathrm{pr}_2 \otimes \mathrm{id}) : (\mathrm{end}(\rho_V) \times \rho_{I[1]}) \otimes \rho_A \Rightarrow \mathrm{end}(\rho_V)$ and $j : \rho_U \to \mathrm{end}(\rho_V) \times \rho_{I[1]}$ the inclusion $[H,\mathscr{M}]$-morphism, we have to show that $\zeta(f) \circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}) = 0$. This is a consequence of item (ii) of Lemma 2.1.26. $\square$

Remark 3.3.3. By Lemma 3.3.2, any element $(L, c) \in \mathrm{con}(V)$ satisfies the condition (3.3.9) for all $a \in A$. In particular, the $\mathbb{Z}$-degree 1 elements $\nabla = (L, 1) \in \mathrm{con}(V)$ satisfy the Leibniz rule with respect to the differential $\mathrm{d}(1)$. Hence, our notion of connections contains the standard notion of connections as distinguished points. It is important to notice that our definition has the advantage that $\mathrm{con}(\rho_V)$ is by construction an object in $[H, \mathscr{M}]$ while the subset of all ordinary connections $\nabla = (L, 1) \in \mathrm{con}(V)$ is just an affine space over the k-module of all $\mathbb{Z}$-degree 1 elements $(L, 0) \in \mathrm{con}(V)$, hence it is not an object in $[H, \mathscr{M}]$. In particular $[H, \mathscr{M}]$ provides a framework in which one can add and rescale connections $\nabla = (L, 1) \in \mathrm{con}(V)$ and $\nabla' = (L', 1) \in \mathrm{con}(V)$ according to

$$c\,\nabla + c'\,\nabla' = c\,(L, 1) + c'\,(L', 1) := (c\,L + c'\,L', c + c')\ , \tag{3.3.10}$$

for $c, c' \in k$ and also act with the triangular quasi-Hopf algebra H according to

$$\rho(h)(\nabla) = \rho(h)(L, 1) := \big(\rho(h)(L), \epsilon(h)1\big)\ , \tag{3.3.11}$$

for $h \in H$. Although in general these are not connections ($\epsilon(h)1$ is in general not equal to 1) it is essential, as we shall see later on, to have these operations for lifting connections to tensor products and internal hom-objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Finally, we prove an important structural result for connections.

Proposition 3.3.4. *Let (ρ_A, d) be any differential calculus in $[H, \mathscr{M}]$ and let ρ_V be any object in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. Then $\mathrm{con}(\rho_V)$ is an $[H, \mathscr{M}]$-subobject of $\mathrm{diff}^1(\rho_V) \times \rho_{I[1]}$.*

Proof. The object $\mathrm{con}(\rho_V)$ is by construction an $[H, \mathscr{M}]$-subobject of $\mathrm{end}(\rho_V) \times \rho_{I[1]}$ and hence the image of $\mathrm{pr}_1 : \mathrm{con}(\rho_V) \Rightarrow \mathrm{end}(\rho_V)$ is an $[H, \mathscr{M}]$-subobject of $\mathrm{end}(\rho_V)$.

We have

$$
\begin{aligned}
[\,\cdot\,,\,\cdot\,]\circ([\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id})\otimes\mathrm{id}_A) &= [\,\cdot\,,\,\cdot\,]\circ(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id})\otimes\mathrm{id}_A)\\
&=0\ ,
\end{aligned}
\tag{3.3.12}
$$

where in the last equality we have used Lemma 2.3.11 for internal homomorphisms in H–Bimod$(A)^{\mathrm{sym}}$ since in fact that $\widehat{l}:\rho_A\Rightarrow\mathrm{end}_A(\rho_V)$ because ρ_A is braided commutative and $\widehat{l}$ is an algebra morphism (cf. Lemma 2.3.8). By using Lemma 3.2.7 this shows that the image of $\mathrm{pr}_1:\mathrm{con}(\rho_V)\Rightarrow\mathrm{end}(\rho_V)$ is an $[H,\mathscr{M}]$-subobject of $\mathrm{diff}^1(\rho_V)$ and hence that $\mathrm{con}(\rho_V)$ is an $[H,\mathscr{M}]$-subobject of $\mathrm{diff}^1(\rho_V)\times\rho_{I[1]}$. $\square$

In other words connections on an object ρ_V in H–Bimod$(A)^{\mathrm{sym}}$ are distinguished differential operators of order 1 in $[H,\mathscr{M}]$.

3.3.2 Connections on tensor products

We now develop a lifting prescription for connections to tensor products of objects in H–Bimod$(A)^{\mathrm{sym}}$. We begin with the observation that a connection on the tensor product module $V\otimes_A W$ ought to be constructed from connections on the components V,W according to a Leibniz rule. Therefore to start with we look for $[H,\mathscr{M}]$ morphisms with source $\mathrm{end}(\rho_V)$ resp. $\mathrm{end}(\rho_W)$ and target $\mathrm{end}(\rho_V\otimes\rho_W)$. The internal tensor product morphism in $[H,\mathscr{M}]$ is an obvious ingredient.

For any two objects ρ_V,ρ_W in H–Bimod$(A)^{\mathrm{sym}}$ there are two $[H,\mathscr{M}]$-morphisms $\boxplus_L$ and $\boxplus_R$ given by the compositions

$$
\begin{tikzcd}
\boxplus_L:\mathrm{end}(\rho_V) \arrow[r, Rightarrow, "\rho^{-1}"] & \mathrm{end}(\rho_V)\otimes\rho_I \arrow[r, Rightarrow, "\mathrm{id}\otimes\eta_{\mathrm{end}(W)}"] & \mathrm{end}(\rho_V)\otimes\mathrm{end}(\rho_W) \arrow[d, Rightarrow, "\odot"] \\
 & & \mathrm{end}(\rho_V\otimes\rho_W) \\
\boxplus_R:\mathrm{end}(\rho_W) \arrow[r, Rightarrow, "\lambda^{-1}"] & \rho_I\otimes\mathrm{end}(\rho_W) \arrow[r, Rightarrow, "\eta_{\mathrm{end}(V)}\otimes\mathrm{id}"] & \mathrm{end}(\rho_V)\otimes\mathrm{end}(\rho_W) \arrow[u, Rightarrow, "\odot"']
\end{tikzcd}
\tag{3.3.13}
$$

The single components of these $[H,\mathscr{M}]$-morphisms are given explicitly by the mappings

$$\boxplus_L(L) = L \circledast 1 \ , \tag{3.3.14}$$

for $L \in \mathrm{end}(V)$ and

$$\boxplus_R(M) = 1 \circledast M \ , \tag{3.3.15}$$

for $M \in \mathrm{end}(W)$.

Definition 3.3.5. For any two objects ρ_V, ρ_W in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ we define the $[H, \mathscr{M}]$-morphism

$$\begin{aligned}\boxplus := \big(\, \boxplus_L \circ \mathrm{pr}_1 \, + \, \boxplus_R \circ \mathrm{pr}_3 \ , \ \mathrm{pr}_2\big) :\\ \big(\mathrm{end}(\rho_V) \times \rho_{I[1]}\big) \times \big(\mathrm{end}(\rho_W) \times \rho_{I[1]}\big) \Longrightarrow \mathrm{end}(\rho_V \otimes \rho_W) \times \rho_{I[1]} \ .\end{aligned} \tag{3.3.16}$$

On the level of elements of objects the single component of the $[H, \mathscr{M}]$-morphism $\boxplus$ in Definition 3.3.5 gives

$$(L, c) \ \boxplus \ (L', c'\,) = \big(L \circledast 1 + 1 \circledast L', c\big) \ , \tag{3.3.17}$$

for any $(L, c) \in \mathrm{end}(V) \times I[1]$ and $(L', c'\,) \in \mathrm{end}(W) \times I[1]$.

In order to prove that $\boxplus$ restricts to connections, i.e. to an $[H, \mathscr{M}]$-morphism $\boxplus : \mathrm{con}(\rho_V) \times \mathrm{con}(\rho_W) \to \mathrm{con}(\rho_V \otimes \rho_W)$, we require the following technical lemma.

Lemma 3.3.6. *Let ρ_A be an object in H–$\mathrm{Alg}^{\mathrm{com}}$ and let ρ_V, ρ_W be any two objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.*

(i) The left ρ_A-action on the internal endomorphism object $\mathrm{end}(\rho_V \otimes \rho_W)$ is given in terms of the left ρ_A-action on $\mathrm{end}(\rho_V)$ by

$$\widehat{l}_{V\otimes W}(\,-\,) = \circledast \circ \big(\widehat{l}_V(\,-\,) \otimes 1\big) \ , \tag{3.3.18}$$

i.e.

$$\widehat{l}_{V\otimes W}(a) = \widehat{l}_V(a) \circledast 1_W \ , \tag{3.3.19}$$

for $a \in A$.

(ii) The commutator on $\mathrm{end}(\rho_V \otimes \rho_W) \times \rho_A$ *is given in terms of the commutator on* $\mathrm{end}(\rho_V) \times \rho_A$ *by*

$$[\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \circ \boxplus \otimes \mathrm{id}_A) = \boxplus_L \circ [\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \otimes \mathrm{id}_A) \ , \tag{3.3.20}$$

i.e.

$$\big[L \circledast 1 + 1 \circledast M \ , a\big] = \big[L \ , a\big] \circledast 1 \ , \tag{3.3.21}$$

for any $a \in A$, $(L, c) \in \mathrm{end}(V) \times I[1]$ and $(M, c') \in \mathrm{end}(W) \times I[1]$.

Proof. For item (i) by the invertibility of the currying bijection it is enough to show that

$$l_{V\otimes W}(-) = \zeta^{-1}(\circledast \circ (\widehat{l}_V(-) \otimes 1_W)) \ , \tag{3.3.22}$$

Using the definition of $\circledast$ given in (2.1.28) and the H-invariance of the unit endomorphism 1_W we have

$$\begin{aligned}
\zeta^{-1}(\circledast \circ (\widehat{l}_V(-) \otimes 1_W)) &= (\mathrm{ev} \otimes \mathrm{id}_W) \circ (\widehat{l}_V \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ \Phi^{-1}_{A,V,W} \\
&= (l_V \otimes \mathrm{id}_W) \circ \Phi^{-1}_{A,V,W} \\
&= l_{V\otimes W} \ .
\end{aligned} \tag{3.3.23}$$

The first equality follows from (2.2.87) using that $\widehat{l}_V$ is H-equivariant, second equality follows from Proposition 2.2.13 (i) and the last equality follows from (2.3.15a).

Item (ii) is a consequence of item (i) and Lemma 3.1.4 as

$$\begin{aligned}\big[L \circledast 1 + 1 \circledast L', a\big] &= \big[L \circledast 1 + 1 \circledast L', \widehat{l}_{V\otimes W}(a)\big] \\ &= \big[L \circledast 1 + 1 \circledast L', \widehat{l}_V(a) \circledast 1\big] = [L, a] \circledast 1 \ , \end{aligned} \tag{3.3.24}$$

for any $a \in A, L \in \mathrm{end}(V)$ and $L' \in \mathrm{end}(W)$, and the assertion follows. □

Proposition 3.3.7. *Let (ρ_A, d) be a differential calculus in $[H, \mathscr{M}]$ and let ρ_V, ρ_W be two objects in H–*$\mathrm{Bimod}(A)^{\mathrm{sym}}$*. Then $\boxplus$ restricts to an $[H, \mathscr{M}]$-morphism*

$$\boxplus : \mathrm{con}(\rho_V) \times \mathrm{con}(\rho_W) \Longrightarrow \mathrm{con}(\rho_V \otimes \rho_W) \ . \tag{3.3.25}$$

Proof. We have to show that the image of $\boxplus : \mathrm{con}(\rho_V) \times \mathrm{con}(\rho_W) \Rightarrow \mathrm{end}(\rho_V \otimes \rho_W) \times \rho_{I[1]}$ is an $[H, \mathscr{M}]$-subobject of $\mathrm{con}(\rho_V \otimes \rho_W)$. Using Lemma 3.3.2 this can be shown by the computation

$$\big[L \circledast 1 + 1 \circledast L', a\big] = [L, a] \circledast 1 = \widehat{l}_V\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \circledast 1 = \widehat{l}_{V\otimes W}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \ , \tag{3.3.26}$$

for all $(L, c) \in \mathrm{con}(V)$, $(L', c') \in \mathrm{con}(W)$ and $a \in A$. In the first equality we used item (ii) and in the last equality item (i) of Lemma 3.3.6. □

The $[H, \mathscr{M}]$-morphism (3.3.25) describes the construction of connections on the object $\rho_V \otimes \rho_W$ but not on the object $\rho_V \otimes_A \rho_W$, which is obtained by using the correct monoidal functor $\otimes_A$ in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. As $\rho_V \otimes_A \rho_W$ can be obtained by taking a quotient of $\rho_V \otimes \rho_W$ (cf. (2.3.21)), we may ask if (3.3.25) induces an $[H, \mathscr{M}]$-morphism with target given by $\mathrm{con}(\rho_V \otimes_A \rho_W)$. For this to hold true, we have to restrict the source of (3.3.25) to the fibred product $\mathrm{con}(\rho_V) \times_{I[1]} \mathrm{con}(\rho_W)$ given by the pullback

$$\begin{tikzcd} \mathrm{con}(\rho_V) \times_{I[1]} \mathrm{con}(\rho_W) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{con}(\rho_W) \arrow[d, Rightarrow, "\mathrm{pr}_2"] \\ \mathrm{con}(\rho_V) \arrow[r, Rightarrow, "\mathrm{pr}_2"'] & \rho_{I[1]} \end{tikzcd} \tag{3.3.27}$$

in the category $[H,\mathscr{M}]$. Then $\mathrm{con}(\rho_V)\times_{I[1]}\mathrm{con}(\rho_W)$ is the $[H,\mathscr{M}]$-subobject of $\mathrm{con}(\rho_V)\times\mathrm{con}(\rho_W)$ with elements given by pairs $((L,c),(L',c'))$ such that $c=c'$. We can now state one of the main results of this section.

Theorem 3.3.8. *Let* (ρ_A,d) *be a differential calculus in* $[H,\mathscr{M}]$ *and let* ρ_V,ρ_W *be two objects in* H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. *Then* $\boxplus$ *induces an* $[H,\mathscr{M}]$*-morphism*

$$\boxplus : \mathrm{con}(\rho_V)\times_{I[1]}\mathrm{con}(\rho_W)\Longrightarrow \mathrm{con}(\rho_V\otimes_A\rho_W)\ . \tag{3.3.28}$$

Proof. Let $((L,c),(L',c))\in\mathrm{con}(V)\times_{I[1]}\mathrm{con}(W)$ be an arbitrary element. Applying $\boxplus$ gives the element

$$\big(L\odot 1+1\odot L',c\big)\in\mathrm{con}(V\otimes W)\subseteq\mathrm{end}(V\otimes W)\times I[1]\ , \tag{3.3.29}$$

where we regard $L\odot 1+1\odot L' : V\otimes W\to V\otimes W$ simply as a k-linear map. We have to prove that $L\odot 1+1\odot L'$ descends to a well-defined k-linear map $L\odot 1+1\odot L' : V\otimes_A W\to V\otimes_A W$ on the quotient (2.3.21). Denoting by $\pi: V\otimes W\to V\otimes_A W$ the quotient map, this amounts to showing that

$$\pi\circ(L\odot 1+1\odot L')\circ(r_V\otimes\mathrm{id}_W)=\pi\circ(L\odot 1+1\odot L')\circ(\mathrm{id}_V\otimes l_W)\circ\Phi_{V,A,W}\ , \tag{3.3.30}$$

The result follows by using (2.2.89) and (2.2.91), the braided symmetry of ρ_V, of $\rho_V\otimes_A\rho_W$ and of ρ_W, the Leibniz rule (3.3.9) for the connections (L,c) and (L',c'), and the equivalence relation (2.3.20) of $\otimes_A$ together with properties of the associator and R-matrix. We also require to use $|L|=|L'|$. □

The following result allows us to consistently lift connections to tensor products of an arbitrary (finite) number of objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Theorem 3.3.9. *Let* (ρ_A,d) *be a differential calculus in* $[H,\mathscr{M}]$ *and let* ρ_V,ρ_W,ρ_X

be three objects in H–Bimod$(A)^{\mathrm{sym}}$. *Then the* $[H,\mathscr{M}]$*-diagram*

$$\begin{tikzcd}
\mathrm{con}(\rho_V) \times_{I[1]} \mathrm{con}(\rho_W) \times_{I[1]} \mathrm{con}(\rho_X) \arrow[r, Rightarrow, "\boxplus\circ(\boxplus\times\mathrm{id})"] \arrow[d, Rightarrow, "\boxplus\circ(\mathrm{id}\times\boxplus)"'] & \mathrm{con}\big((\rho_V \otimes_A \rho_W) \otimes_A \rho_X\big) \arrow[dl, Rightarrow, "\Phi\circ(\,\cdot\,)\circ\Phi^{-1}"] \\
\mathrm{con}\big(\rho_V \otimes_A (\rho_W \otimes_A \rho_X)\big) &
\end{tikzcd} \tag{3.3.31}$$

commutes.

Proof. Let $\big((L,c),(L',c),(L'',c)\big) \in \mathrm{con}(V) \times_{I[1]} \mathrm{con}(W) \times_{I[1]} \mathrm{con}(X)$ be an arbitrary element. Applying $\boxplus \circ (\boxplus \times \mathrm{id})$ yields

$$\boxplus \circ(\boxplus \times \mathrm{id}) \Big(\big((L,c),(L',c),(L'',c)\big)\Big) = \\ \big((L \odot 1) \odot 1 + (1 \odot L') \odot 1 + (1 \odot 1) \odot L'', c\big) \tag{3.3.32a}$$

while applying $\boxplus \circ (\mathrm{id} \times \boxplus)$ yields

$$\boxplus \circ(\mathrm{id} \times \boxplus) \Big(\big((L,c),(L',c),(L'',c)\big)\Big) = \\ \big(L \odot (1 \odot 1) + 1 \odot (L' \odot 1) + 1 \odot (1 \odot L''), c\big) \ . \tag{3.3.32b}$$

The assertion then follows by using Proposition 2.2.30. $\square$

3.3.3 Connections on internal homomorphisms

We now develop a lifting prescription for connections to the internal hom-objects in H–Bimod$(A)^{\mathrm{sym}}$. This is an important construction in differential geometry because if V is an object in H–Bimod$(A)^{\mathrm{sym}}$ then the internal hom-object $\hom_A(V,A)$ is the dual object to V. One may take V to be the module of sections of the tangent bundle of a manifold and then $\hom_A(V,A)$ is the module of sections of the bundle of one-forms. It is of importance in differential geometry to know how to construct a connection on the one-forms from a connection on the tangent bundle (see e.g. [60]).

Let (ρ_A, d) be a differential calculus in $[H,\mathscr{M}]$ and ρ_V, ρ_W any two objects in

H–Bimod$(A)^{\mathrm{sym}}$. We require to construct a connection on $\hom_A(\rho_V, \rho_W)$ from connections on ρ_V and ρ_W. Similarly to the lifting of connections to tensor products we start by looking for $[H, \mathscr{M}]$ morphisms with source $\mathrm{end}(\rho_V)$ resp. $\mathrm{end}(\rho_W)$ and target $\mathrm{end}(\hom(\rho_V, \rho_W))$. We observe that it is possible to take the internal tensor product of $\mathrm{end}(\rho_V)$ resp. $\mathrm{end}(\rho_W)$ with $\hom(\rho_V, \rho_W)$ in $[H, \mathscr{M}]$ and then by using the currying map for internal homomorphisms in $[H, \mathscr{M}]$ obtain two $[H, \mathscr{M}]$-morphisms with the desired source and target objects which we denote by:

$$\mathscr{L} := \zeta(\bullet) : \mathrm{end}(\rho_W) \Longrightarrow \mathrm{end}(\hom(\rho_V, \rho_W)) \ , \tag{3.3.33a}$$

$$\mathscr{R} := \zeta(\bullet \circ \tau) : \mathrm{end}(\rho_V) \Longrightarrow \mathrm{end}(\hom(\rho_V, \rho_W)) \ . \tag{3.3.33b}$$

Definition 3.3.10. For any two objects ρ_V, ρ_W in H–Bimod$(A)^{\mathrm{sym}}$ we define the $[H, \mathscr{M}]$-morphism

$$\begin{aligned} &\mathrm{ad}_\bullet := \big(\mathscr{L} \circ \mathrm{pr}_1 \ - \ \mathscr{R} \circ \mathrm{pr}_3 \ , \ \mathrm{pr}_2\big) : \\ &\qquad \big(\mathrm{end}(\rho_W) \times \rho_{I[1]}\big) \times \big(\mathrm{end}(\rho_V) \times \rho_{I[1]}\big) \Longrightarrow \mathrm{end}(\hom(\rho_V, \rho_W)) \times \rho_{I[1]} \ , \end{aligned} \tag{3.3.34}$$

On the level of elements of objects the single component of the $[H, \mathscr{M}]$-morphism $\mathrm{ad}_\bullet$ in Definition 3.3.10 gives

$$\mathrm{ad}_\bullet\big((L', c'\,), (L, c)\big) = \big(\mathscr{L}(L'\,) - \mathscr{R}(L), c'\,\big) \ , \tag{3.3.35}$$

for any $(L, c) \in \mathrm{end}(V) \times I[1]$ and $(L', c'\,) \in \mathrm{end}(W) \times I[1]$.

We shall require the following two technical lemmas.

Lemma 3.3.11. *Let ρ_A be an object in H–*Alg$^{\mathrm{com}}$ *and ρ_V, ρ_W any two objects in H–*Bimod$(A)^{\mathrm{sym}}$. *One has*

$$\widehat{l}_{\hom(V,W)} = \mathscr{L} \circ \widehat{l}_W \ , \tag{3.3.36}$$

Proof. Recalling (2.3.29) and using naturality of the currying bijection yields

$$
\begin{aligned}
\widehat{l}_{\mathrm{hom}(V,W)} &= \zeta\big(\,\bullet \circ(\widehat{l}_W \otimes \mathrm{id})\big)\\
&= \zeta\big(\mathrm{Hom}_{[H,\mathscr{M}]}(\widehat{l}_W^{\,\mathrm{op}} \otimes \mathrm{id}^{\mathrm{op}}, \mathrm{id})(\bullet)\big)\\
&= \mathrm{Hom}_{[H,\mathscr{M}]}\big(\widehat{l}_W^{\,\mathrm{op}}, \mathrm{hom}(\mathrm{id}^{\mathrm{op}}, \mathrm{id})\big)\big(\zeta(\bullet)\big)\\
&= \zeta(\bullet) \circ \widehat{l}_W = \mathscr{L} \circ \widehat{l}_W \ ,
\end{aligned}
\tag{3.3.37}
$$

□

Lemma 3.3.12. *Let ρ_V, ρ_W be any two objects in H–*$\mathrm{Bimod}(A)^{\mathrm{sym}}$*. Then*

$$
\bullet \circ (\mathscr{L} \otimes \mathscr{L}) = \mathscr{L} \circ \bullet \ , \tag{3.3.38a}
$$

$$
\bullet \circ (\mathscr{R} \otimes \mathscr{L}) = \bullet \circ (\mathscr{L} \otimes \mathscr{R}) \circ \tau \ , \tag{3.3.38b}
$$

Proof. By invertibility of the natural currying bijections, equation (3.3.38a) holds true if $\zeta^{-1}(\bullet \circ (\mathscr{L} \otimes \mathscr{L})) = \zeta^{-1}(\mathscr{L} \circ \bullet)$ as morphisms from $(\mathrm{end}(\rho_W) \otimes \mathrm{end}(\rho_W)) \otimes \mathrm{hom}(\rho_V, \rho_W) \Rightarrow \mathrm{hom}(\rho_V, \rho_W)$. This can be shown by using Lemma 2.1.26 (i) and the calculation

$$
\begin{aligned}
\zeta^{-1}(\bullet \circ (\mathscr{L} \otimes \mathscr{L})) &= \mathrm{ev} \circ (\bullet \circ (\zeta(\bullet) \otimes \zeta(\bullet)) \otimes \mathrm{id})\\
&= \mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \big((\zeta(\bullet) \otimes \zeta(\bullet)) \otimes \mathrm{id}\big) \circ \Phi\\
&= \bullet \circ (\mathrm{id} \otimes \bullet) \circ \Phi \ ,
\end{aligned}
\tag{3.3.39}
$$

where we have used the H-equivariance of $\zeta(\bullet)$ in the third step, and

$$
\begin{aligned}
\zeta^{-1}(\mathscr{L} \circ \bullet) &= \mathrm{ev} \circ \big((\zeta(\bullet) \circ \bullet) \otimes \mathrm{id}\big)\\
&= \bullet \circ (\bullet \otimes \mathrm{id}) \ .
\end{aligned}
\tag{3.3.40}
$$

These equations agree due to the weak associativity of the internal composition (cf. Proposition 2.2.13 (iii)). The equality (3.3.38b) can be shown similarly. By a similar calculation to (3.3.39) we have

$$\begin{aligned}\zeta^{-1}(\bullet\circ(\mathscr{R}\otimes\mathscr{L})) &= (\bullet\circ\tau)\circ(\mathrm{id}\otimes\bullet)\circ\Phi\\ &= \bullet\circ(\bullet\otimes\mathrm{id})\circ\tau_{1,23}\circ\Phi\\ &= \bullet\circ(\mathrm{id}\otimes\bullet)\circ\Phi\circ\tau_{1,23}\circ\Phi\ ,\end{aligned}\tag{3.3.41}$$

using weak associativity of the internal composition $\bullet$ in the final step, and

$$\begin{aligned}\zeta^{-1}(\bullet\circ(\mathscr{L}\otimes\mathscr{R})\circ\tau) &= \bullet\circ(\mathrm{id}\otimes(\bullet\circ\tau))\circ\Phi\circ(\tau\otimes\mathrm{id})\\ &= \bullet\circ(\mathrm{id}\otimes\bullet)\circ\tau_{23}\circ\Phi\circ\tau_{12}\\ &= \bullet\circ(\mathrm{id}\otimes\bullet)\circ\Phi\circ\tau_{1,23}\circ\Phi\ ,\end{aligned}\tag{3.3.42}$$

using (2.1.103b) in the final step. □

Proposition 3.3.13. *Let (ρ_A,d) be a differential calculus in $[H,\mathscr{M}]$ and let ρ_V,ρ_W be two objects in H–*Bimod$(A)^{\mathrm{sym}}$*. Then* $\mathrm{ad}_\bullet$ *restricts to an $[H,\mathscr{M}]$-morphism*

$$\mathrm{ad}_\bullet:\mathrm{con}(\rho_W)\times\mathrm{con}(\rho_V)\Longrightarrow\mathrm{con}(\mathrm{hom}(\rho_V,\rho_W))\ .\tag{3.3.43}$$

Proof. It must be shown that the target of the $[H,\mathscr{M}]$-morphism $\mathrm{ad}_\bullet:\mathrm{con}(\rho_W)\times\mathrm{con}(\rho_V)\Rightarrow\mathrm{end}(\mathrm{hom}(\rho_V,\rho_W))\times\rho_{I[1]}$ is an $[H,\mathscr{M}]$-subobject of $\mathrm{con}(\mathrm{hom}(\rho_V,\rho_W))$. Using Lemma 3.3.2 this can be shown by the computation

$$\begin{aligned}\big[\mathscr{L}(L')-\mathscr{R}(L),a\big] &= \big[\mathscr{L}(L')-\mathscr{R}(L),\widehat{l}_{\mathrm{hom}(V,W)}(a)\big]\\ &= \big[\mathscr{L}(L')-\mathscr{R}(L),\mathscr{L}(\widehat{l}_W(a))\big]\\ &= \mathscr{L}\big([L',a]\big)\\ &= \mathscr{L}\big(\widehat{l}_W(\mathrm{ev}(\mathrm{d}(c')\otimes a))\big)\\ &= \widehat{l}_{\mathrm{hom}(V,W)}\big(\mathrm{ev}(\mathrm{d}(c')\otimes a)\big)\ ,\end{aligned}\tag{3.3.44}$$

for all $(L',c')\in\mathrm{con}(W)$, $(L,c)\in\mathrm{con}(V)$ and $a\in A$. In the second and last equality we have used Lemma 3.3.11 and in the third equality we have used Lemma 3.3.12. $\square$

Restricting the source of $\mathrm{ad}_\bullet$ to the fibred product $\mathrm{con}(\rho_W)\times_{I[1]}\mathrm{con}(\rho_V)$ we obtain a lifting prescription of connections to the internal hom-objects $\hom_A(\rho_V,\rho_W)$ in the category H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$.

Theorem 3.3.14. *Let* (ρ_A,d) *be a differential calculus in* $[H,\mathscr{M}]$ *and let* ρ_V,ρ_W *be two objects in* H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$*. Then* $\mathrm{ad}_\bullet$ *induces an* $[H,\mathscr{M}]$*-morphism*

$$\mathrm{ad}_\bullet : \mathrm{con}(\rho_W)\times_{I[1]}\mathrm{con}(\rho_V)\Longrightarrow \mathrm{con}(\hom_A(\rho_V,\rho_W)) ~. \tag{3.3.45}$$

Proof. Let $((L',c),(L,c))\in\mathrm{con}(W)\times_{I[1]}\mathrm{con}(V)$ be an arbitrary element. Applying $\mathrm{ad}_\bullet$ gives the element

$$\big(\mathscr{L}(L')-\mathscr{R}(L),c\big)\in\mathrm{con}(\hom(V,W))\subseteq\mathrm{end}(\hom(V,W))\times I[1] ~, \tag{3.3.46}$$

where we regard $\mathscr{L}(L')-\mathscr{R}(L):\hom(V,W)\to\hom(V,W)$ as a k-linear map. We have to prove that $\mathscr{L}(L')-\mathscr{R}(L)$ restricts to a k-linear map $\mathscr{L}(L')-\mathscr{R}(L):\hom_A(V,W)\to\hom_A(V,W)$ on the k-submodules $\hom_A(V,W)\subseteq\hom(V,W)$ given in (2.3.36). This amounts to showing that

$$[\,\cdot\,,\,\cdot\,]\circ\big((\mathscr{L}(L')-\mathscr{R}(L))\otimes\mathrm{id}_A\big)=0 ~, \tag{3.3.47}$$

(cf. Lemma 2.3.11). We have

$$\begin{aligned}
&[\,\cdot\,,\,\cdot\,]\circ\big(\mathscr{L}(L')\otimes\mathrm{id}_A\big)=[\,\cdot\,,\,\cdot\,]\circ(\bullet\otimes\mathrm{id}_A)\circ(\zeta(L')\otimes\mathrm{id}_A)\\
&=\bullet\circ\Big(\big(\mathrm{id}\otimes[\,\cdot\,,\,\cdot\,]\big)+\big([\,\cdot\,,\,\cdot\,]\otimes\mathrm{id}\big)\circ\Phi^{-1}\circ\big(\mathrm{id}\otimes\tau\big)\Big)\circ\Phi\circ(\zeta(L')\otimes\mathrm{id}_A)\\
&=\bullet\circ\big(\widehat{l}\circ\mathrm{ev}\circ\big(\mathrm{d}(c)\otimes\mathrm{id}_A\big)\otimes\mathrm{id}\big)\circ\tau\circ\big(\zeta(\mathrm{id})\otimes\mathrm{id}_A\big)\\
&=\bullet\circ(\mathrm{id}\otimes\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}(c)\otimes\mathrm{id}_A))\circ(\zeta(\mathrm{id})\otimes\mathrm{id}_A) ~,
\end{aligned} \tag{3.3.48}$$

where in the second equality we have used the biderivation property of the commutator (2.1.41), the first term of the second equality vanishes due to the braided symmetry of the H–Bimod$(A)^{\text{sym}}$-object $\hom_A(\rho_V, \rho_W)$, the third equality then follows from the Leibniz rule for the connection (L', c) together with (3.3.11) and the property $(\epsilon \otimes 1 \otimes 1)(\phi) = 1 \otimes 1$, the last equality again follows from the braided symmetry of the H–Bimod$(A)^{\text{sym}}$-object $\hom_A(\rho_V, \rho_W)$ together with the triangularity of the R-matrix. On the other hand we have

$$
\begin{aligned}
&[\,\cdot\,,\,\cdot\,] \circ \big(\mathscr{R}(L\,) \otimes \mathrm{id}_A\big) = [\,\cdot\,,\,\cdot\,] \circ (\bullet \circ \tau \otimes \mathrm{id}_A) \circ (\zeta(L\,) \otimes \mathrm{id}_A) \\
&= \bullet \circ \Big(\big(\mathrm{id} \otimes [\,\cdot\,,\,\cdot\,]\big) + \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \Phi^{-1} \circ \big(\mathrm{id} \otimes \tau\big)\Big) \circ \Phi \circ (\tau \otimes \mathrm{id}_A) \circ (\zeta(L\,) \otimes \mathrm{id}_A) \\
&= \bullet \circ \big(\mathrm{id} \otimes [\,\cdot\,,\,\cdot\,]\big) \circ \Phi \circ (\tau \otimes \mathrm{id}_A) \circ (\zeta(L\,) \otimes \mathrm{id}_A) \\
&= \bullet \circ \Big(\mathrm{id} \otimes \widehat{l} \circ \mathrm{ev} \circ (\mathrm{d}(c) \otimes \mathrm{id}_A)\Big) \circ (\zeta(\mathrm{id}) \otimes \mathrm{id}_A)\ ,
\end{aligned} \tag{3.3.49}
$$

where we again use the biderivation property of the commutator (2.1.41) in the second equality and in this case the second term vanishes due to the braided symmetry of the H–Bimod$(A)^{\text{sym}}$-object $\hom_A(\rho_V, \rho_W)$, the fourth equality follows from the Leibniz rule for the connection (L, c) together with (3.3.11) and the properties $(\epsilon \otimes 1 \otimes 1)(\phi) = 1 \otimes 1$ and $(\epsilon \otimes 1)(R) = 1$. This completes the proof. □

3.3.4 Cochain twisting of connections

The cochain twist deformation quantization functor preserves connections.

Proposition 3.3.15. *Let (ρ_A, d) be any differential calculus in $[H, \mathscr{M}]$, let ρ_V be any object in H–*Bimod$(A)^{\text{sym}}$ *and F any cochain twisting element based on H. Then the coherence map $\gamma \times \psi : \mathrm{end}_F(\mathcal{F}(\rho_V)) \times \rho_{I_F[1]} \Rightarrow \mathcal{F}(\mathrm{end}(\rho_V)) \times \mathcal{F}(\rho_{I[1]})$ restricts to an $[H_F, \mathscr{M}]$-isomorphism*

$$
\gamma \times \psi : \mathrm{con}_F(\mathcal{F}(\rho_V)) \Longrightarrow \mathcal{F}(\mathrm{con}(\rho_V))\ . \tag{3.3.50}
$$

Proof. The proof follows that of Proposition 3.2.4 and it requires showing commutativity of the diagram

$$
\begin{tikzcd}
\mathrm{end}_F(\mathcal{F}(\rho_V)) \times \rho_{I_F[1]} \arrow[r, shift left, "\zeta_F([\,\cdot\,,\cdot\,]_F\circ(\mathrm{pr}_1\otimes_F\mathrm{id}))"] \arrow[r, shift right, "\zeta_F(\widehat{l}_F\circ\mathrm{ev}_F\circ(\mathrm{d}_F\otimes_F\mathrm{id})\circ(\mathrm{pr}_2\otimes_F\mathrm{id}))"'] \arrow[dd, Rightarrow, "\gamma\times\psi"'] & \mathrm{hom}_F(\mathcal{F}(\rho_A), \mathrm{end}_F(\mathcal{F}(\rho_V))) \arrow[d, Rightarrow, "\gamma\circ(\,\cdot\,)"] \\
 & \mathrm{hom}_F(\mathcal{F}(\rho_A), \mathcal{F}(\mathrm{end}(\rho_V))) \arrow[d, Rightarrow, "\gamma"] \\
\mathcal{F}(\mathrm{end}(\rho_V)) \times \mathcal{F}(\rho_{I[1]}) \arrow[r, shift left, "\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id})))"] \arrow[r, shift right, "\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id})))"'] & \mathcal{F}\big(\mathrm{hom}(\rho_A, \mathrm{end}(\rho_V))\big)
\end{tikzcd}
\tag{3.3.51}
$$

in $[H_F, \mathscr{M}]$. The upper set of arrows commute by the same calculation as in (3.2.14a) (equivalently (2.3.60)). For the lower set of arrows, we have

$$
\begin{aligned}
\gamma\circ\gamma\circ(\,\cdot\,)\circ\zeta_F(\widehat{l}_F\circ\mathrm{ev}_F\circ(\mathrm{d}_F\otimes_F\mathrm{id})) &= \gamma\circ\gamma\circ(\,\cdot\,)\circ\zeta_F(\widehat{l}_F\circ\mathrm{ev}_F)\circ\mathrm{d}_F \\
&= \gamma\circ\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}))\circ\gamma^{-1}\circ\mathcal{F}(\mathrm{d})\circ\psi \\
&= \mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})))\circ\psi\ .
\end{aligned}
\tag{3.3.52}
$$

The first equality follows from the H_F-equivariance of d_F, the second equality follows from (3.2.14b) and (3.2.31), and the final equality follows from the H_F-equivariance of $\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}))$. □

Remark 3.3.16. The isomorphism in Proposition 3.3.15 extends to an isomorphism between connections on tensor product objects and also on internal hom-objects in H_F–Bimod$(A_F)^{\mathrm{sym}}$. For tensor product objects, diagram (3.3.51) contains additionally in each vertical arrow the composition with the isomorphism φ and, likewise for the internal hom-objects, the isomorphism $\gamma\circ(\cdot)$. The H_F-equivariance of morphisms in diagram (3.3.51) ensures that these additional isomorphisms cancel each other out.

3.4 Curvature

We develop the notion of curvature of connections on objects in H–Bimod$(A)^{\mathrm{sym}}$ and compute explicitly the curvatures of tensor product connections given by the construction in Theorem 3.3.8.

3.4.1 Definition and properties

For any object ρ_V in H–Bimod$(A)^{\mathrm{sym}}$, we define the $[H,\mathscr{M}]$-morphism

$$[\![\,\cdot\,,\,\cdot\,]\!] := [\,\cdot\,,\,\cdot\,] \circ \big(\mathrm{pr}_1 \otimes \mathrm{pr}_3\big) : (\mathrm{end}(\rho_V) \times \rho_{I[1]}) \otimes (\mathrm{end}(\rho_V) \times \rho_{I[1]}) \Longrightarrow \mathrm{end}(\rho_V)\ .$$

On the level of elements

$$[\![(L,c)\ ,\ (L',c'\,)]\!] = [L,L'\,]\ , \tag{3.4.1}$$

for all (L,c) and $(L',c'\,)$ in $\mathrm{end}(V) \times I[1]$.

Lemma 3.4.1. *Let H be a triangular quasi-Hopf algebra. Let (ρ_A, d) be a differential calculus in $[H,\mathscr{M}]$ and ρ_V any object in H–*Bimod$(A)^{\mathrm{sym}}$*. Then (3.4.1) restricts to an $[H,\mathscr{M}]$-morphism*

$$[\![\,\cdot\,,\,\cdot\,]\!] : \mathrm{con}(\rho_V) \otimes \mathrm{con}(\rho_V) \Longrightarrow \mathrm{end}_A(\rho_V)\ . \tag{3.4.2}$$

Proof. By Lemma 2.3.11 it is sufficient to show that

$$\big[[\![(L,c),(L',c'\,)]\!]\ ,\ a\big] = \big[[L,L'\,],a\big] = 0\ , \tag{3.4.3}$$

for all $(L,c),(L',c'\,) \in \mathrm{con}(V)$ and $a \in A$. We have

$$\begin{aligned}
[\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}_A\big) &= [\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}_A\big) \circ (-\tau \circ \Phi\ -\ \Phi^{-1} \circ \tau) \\
&= [\,\cdot\,,\,\cdot\,] \circ \big(\mathrm{id} \otimes \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}_A)\big) \cdot 2 \\
&= \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}_A) \circ \big(\mathrm{id} \otimes \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}_A)\big) \cdot 2 \\
&= \mathrm{ev} \circ (\bullet \otimes \mathrm{id}) \circ (\mathrm{d} \otimes \mathrm{d} \otimes \mathrm{id}_A) \cdot 2 \\
&= 0\ ,
\end{aligned} \tag{3.4.4}$$

where the first equality follows from the braided Jacobi identity in Proposition 2.1.13, the second equality follows from Lemma 3.3.2, the H-invariance of $\mathrm{d} : \rho_{I[1]} \Rightarrow \mathrm{der}(\rho_A)$ together with the normalization $(\epsilon \otimes \epsilon \otimes \mathrm{id})(\phi) = 1$ of the associator and of the R-matrix $(\epsilon \otimes \mathrm{id})(R) = 1$ and also the braided antisymmetry of the internal commutator in Proposition 2.1.13 together with the triangularity of the R-matrix and $|L| = 1 = |L'|$, the third equality follows from Lemma 3.3.2, the fourth equality follows from Proposition 2.2.13 (ii), the last equality follows from the nilpotency of d from Definition 3.2.11. □

With these techniques we can now define the curvature of a connection. Since the curvature is supposed to be quadratic in the connections, we cannot realize the assignment of curvatures as an $[H, \mathscr{M}]$-morphism. We shall employ the following element-wise left-right symmetric definition.

Definition 3.4.2 (Curvature)**.** Let (ρ_A, d) be a differential calculus in $[H, \mathscr{M}]$ and let ρ_V be an object in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. The *curvature of a connection* $\nabla := (L, 1) \in \mathrm{con}(V)$ is the element

$$\mathrm{Curv}(\nabla) := [\![\nabla, \nabla]\!] \in \mathrm{end}_A(V) \ . \tag{3.4.5}$$

Remark 3.4.3. Given any connection $\nabla := (L, 1) \in \mathrm{con}(V)$, we can define the Bianchi tensor corresponding to ∇ as

$$\mathrm{Bianchi}(\nabla) := \mathrm{ev}\big(\mathrm{ad}_\bullet(\nabla, \nabla) \otimes \mathrm{Curv}(\nabla)\big) \in \mathrm{end}_A(V) \ . \tag{3.4.6}$$

In contrast to the situation in classical differential geometry, here the Bianchi tensor in general does not vanish. Hence, it may be interpreted as a measure of the noncommutativity and nonassociativity of ρ_A, ρ_V and ∇.

Remark 3.4.4. Flat connections are in general not twisted to flat connections unless the connection is invariant under the action of the cochain twist, since for a connection $\nabla = (L, c) \in \mathrm{con}(V)$ and using (2.2.108b) and the isomorphism (3.3.50)

we have

$$\mathrm{Curv}_F((\gamma^{-1} \times \psi^{-1})(\nabla)) = \gamma^{-1} \circ \mathcal{F}([\,\cdot\,,\,\cdot\,]) \circ \varphi \circ (L \otimes L) \ . \tag{3.4.7}$$

which may not vanish even if $\mathrm{Curv}(\nabla) = [L\,,\,L] = 0$.

Finally, we observe an additive property of the curvature of the tensor product connections constructed in Theorem 3.3.8.

Proposition 3.4.5. *Let H be a triangular quasi-Hopf algebra, (ρ_A, d) a differential calculus in $[H, \mathscr{M}]$ and ρ_V, ρ_W two objects in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$. Given any two connections $\nabla_V := (L, 1) \in \mathrm{con}(V)$ and $\nabla_W := (L', 1) \in \mathrm{con}(W)$, the curvature of their sum satisfies*

$$\mathrm{Curv}(\nabla_V \boxplus \nabla_W) = \mathrm{Curv}(\nabla_V) \circledast 1 + 1 \circledast \mathrm{Curv}(\nabla_W) \ . \tag{3.4.8}$$

Proof. The proof follows from a simple calculation

$$\begin{aligned} \mathrm{Curv}(\nabla_V \boxplus \nabla_W) &= [L \circledast 1 + 1 \circledast L', L \circledast 1 + 1 \circledast L'\,] \\ &= [L, L] \circledast 1 + 1 \circledast [L', L'\,] \\ &= \mathrm{Curv}(\nabla_V) \circledast 1 + 1 \circledast \mathrm{Curv}(\nabla_W) \ , \end{aligned} \tag{3.4.9}$$

where we have used the properties in Lemma 3.1.4. □

3.5 Summary

In this chapter we have formulated the notions of differential calculus, connection and curvature in the representation category of an arbitrary triangular quasi-Hopf algebra H on bounded $\mathbb{Z}$-graded k-modules. We have made use of equalisers in the category $[H, \mathscr{M}]$ to formulate the notions of derivations on ρ_A, and differential operators and, because the unit object for the monoidal structure in H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ is a graded differential algebra ρ_A in $[H, \mathscr{M}]$ also connections on symmetric bimodules over ρ_A as subobjects of internal endomorphism objects in the category

$[H, \mathscr{M}]$. Most importantly we have found appropriate morphisms to lift connections in $[H, \mathscr{M}]$ to tensor products and internal hom-objects in the closed braided monoidal category H–Bimod$(A)^{\mathrm{sym}}$. We have also shown that cochain twist quantisation preserves structurally all these constructions by the same isomorphism which preserves the internal endomorphism objects in H–Bimod$(A)^{\mathrm{sym}}$. In the next chapter we apply this framework to obtain explicit expressions for connections and their curvatures on noncommutative and nonassociative vector bundles in the simplest example of cochain twist deformations of trivial vector bundles over noncommutative and nonassociative spaces.

Chapter 4

Working with Nonassociative Geometry and Field Theory

This chapter is based on the last section of [34] and [35].

In the previous chapter we described notions of differential geometry in the representation category of an arbitrary triangular quasi-Hopf algebra. The categorical formalism enabled us to make structurally correct definitions for the notions of connections together with their tensor product structure in particular. This chapter is divided into two parts. In Section 4.1 we apply the constructions in Chapter 2 to the concrete examples of deformation quantization of G-equivariant vector bundles over G-manifolds. In particular we construct concrete examples for the categories H–$\mathrm{Alg}^{\mathrm{com}}$ and H–$\mathrm{Bimod}(A)^{\mathrm{sym}}$ for a given braided commutative algebra $\rho_A \in H$–$\mathrm{Alg}^{\mathrm{com}}$ starting from ordinary differential geometry. In these examples the algebras ρ_A and bimodules ρ_V are commutative, i.e. braided commutative with respect to the trivial R-matrix $R = 1 \otimes 1$. Deformation quantization by cochain twists then leads to examples of noncommutative and also nonassociative algebras and bimodules. In Sections 4.2 - 4.4 we consider concrete realizations of the notions of geometry developed in Chapter 3 in the simplest example of cochain twist deformations of *trivial* vector bundles over noncommutative and nonassociative spaces with the R-flux and Q-flux compactification of closed string theory as the main motivating examples. We conclude by providing physically viable action functionals for Yang-Mills theory and Einstein-Cartan gravity on such noncommutative and nonassociative spaces, as first steps towards more elaborate models relevant to non-geometric flux deformations of geometry in closed string theory.

4.1 Quantization of equivariant vector bundles

Let Man denote the category of C^∞, finite-dimensional, Hausdorff and second countable manifolds with smooth maps.

Recall that associated to any manifold M in Man is the Lie algebra $\mathrm{Vec}(M)$ of vector fields on M (with Lie bracket $[\,\cdot\,,\,\cdot\,]$ given by the vector field commutator), which plays the role of the infinitesimal diffeomorphisms of M. This Lie algebra gives rise to a Hopf algebra $U\mathrm{Vec}(M)$, the universal enveloping algebra of $\mathrm{Vec}(M)$, which is characterized as follows: As an algebra, $U\mathrm{Vec}(M)$ is the free unital algebra generated by $\mathrm{Vec}(M)$ modulo the relations $v\,w - w\,v = [v,w]$, for all $v,w \in \mathrm{Vec}(M)$. The coproduct Δ, counit ϵ and antipode S on $U\mathrm{Vec}(M)$ are defined on generators by

$$\Delta(v) = v\otimes 1 + 1\otimes v \ , \quad \Delta(1) = 1\otimes 1 \ , \tag{4.1.1a}$$

$$\epsilon(v) = 0 \ , \quad \epsilon(1) = 1 \ , \tag{4.1.1b}$$

$$S(v) = -v \ , \quad S(1) = 1 \ , \tag{4.1.1c}$$

for all $v \in \mathrm{Vec}(M)$. The maps Δ and ϵ are extended as algebra homomorphisms and S as an anti-algebra homomorphism to all of $U\mathrm{Vec}(M)$. There is then an exponential map $\exp : \mathrm{Vec}(M) \to G$ where G is a (complex) Lie group.

Let us fix any (complex) Lie group G and denote its Lie algebra by $\mathfrak{g}$. We view G as a one-object category (cf. Definition A.3.4 for the definition of a group as a one-object category) and consider the functor category $[G,\mathsf{Man}]$ defined as follows: The objects in $[G,\mathsf{Man}]$ are functors

$$\rho_M : G \longrightarrow \mathsf{Man} \ , \tag{4.1.2}$$

with $\rho_M(*) = \underline{M}$ a manifold and $\rho_M(g) := \triangleright_M(g,\,\cdot\,)$ where $\triangleright_M(\,\cdot\,,\,\cdot\,) : G\times\underline{M} \to \underline{M}$ is a smooth left G-action on $\underline{M}$. The morphisms in $[G,\mathsf{Man}]$ are natural transformations

$$f : \rho_M \Longrightarrow \rho_N \ , \tag{4.1.3}$$

for some manifolds $\underline{M}, \underline{N}$. The naturality condition implies that f is a G-equivariant smooth map, i.e. a smooth map (denoted by the same symbol) $f : \underline{M} \to \underline{N}$, such that the diagram

$$\begin{tikzcd} \underline{M} \arrow[r, "f"] \arrow[d, "\rho_M(g)"'] & \underline{N} \arrow[d, "\rho_N(g)"] \\ \underline{M} \arrow[r, "f"'] & \underline{N} \end{tikzcd} \tag{4.1.4}$$

commutes in Man for any $g \in G$.

Next we view the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ as a one-object category with morphisms the elements of $U\mathfrak{g}$ and composition given by the multiplication in $U\mathfrak{g}$, and consider the functor category $[U\mathfrak{g}, \mathscr{M}]$ defined as follows: The objects in $[U\mathfrak{g}, \mathscr{M}]$ are functors

$$\rho_V : U\mathfrak{g} \longrightarrow \mathscr{M} \ . \tag{4.1.5}$$

with $\rho_V(*) = \underline{V}$ a $\mathbb{C}$-module and $\rho_V(\xi) := \triangleright_V(\xi, \text{-})$ where $\triangleright_V(\text{-}, \cdot\,) : U\mathfrak{g} \times \underline{V} \to \underline{V}$ is a left $U\mathfrak{g}$-action on $\underline{V}$. We recall that the category of $\mathbb{C}$-modules $\mathscr{M}$ is a braided monoidal category with monoidal functor $\otimes_{\mathbb{C}}$ the tensor product of $\mathbb{C}$-modules and unit $\mathbb{C}$. The category $[U\mathfrak{g}, \mathscr{M}]$ is also a monoidal category. The unit object is given by $\rho_{\mathbb{C}}(*) = \mathbb{C}$ and $\rho_{\mathbb{C}}(\xi) = \triangleright_{\mathbb{C}}(\xi, -)$ where $\triangleright_{\mathbb{C}}(\xi, c) = \epsilon(\xi)c$ is given by the counit ϵ in $U\mathfrak{g}$, and the monoidal structure $\otimes_{\rho_{\mathbb{C}}} : [U\mathfrak{g}, \mathscr{M}] \times [U\mathfrak{g}, \mathscr{M}] \to [U\mathfrak{g}, \mathscr{M}]$ is given by $\rho_V \otimes_{\rho_{\mathbb{C}}} \rho_W(*) = \underline{V} \otimes_{\mathbb{C}} \underline{W}$ and $\rho_V \otimes_{\rho_{\mathbb{C}}} \rho_W(\xi) = \triangleright_{\underline{V}\otimes_{\mathbb{C}}\underline{W}}(\xi, -)$ where $\triangleright_{\underline{V}\otimes_{\mathbb{C}}\underline{W}}(\xi, v \otimes_{\mathbb{C}} w) = (\xi_{(1)} \triangleright_V v) \otimes_{\mathbb{C}} (\xi_{(2)} \triangleright_W w)$ (summation understood) is given by the coproduct Δ in $U\mathfrak{g}$. Because ϵ and Δ are algebra morphisms, these are valid $U\mathfrak{g}$-actions. The associator and unitors are trivial since $U\mathfrak{g}$ is a Hopf algebra. The morphisms in $[U\mathfrak{g}, \mathscr{M}]$ are natural transformations

$$f : \rho_V \Longrightarrow \rho_W \ , \tag{4.1.6}$$

for some $\mathbb{C}$-modules $\underline{V}, \underline{W}$. The naturality condition implies that morphisms in $[U\mathfrak{g}, \mathscr{M}]$ are $U\mathfrak{g}$-equivariant $\mathbb{C}$-module maps. Two morphisms of particular interest are the product $\mu_A : \rho_A \otimes_{\rho_{\mathbb{C}}} \rho_A \Rightarrow \rho_A$ and unit $\eta_A : \rho_{\mathbb{C}} \Rightarrow \rho_A$ which endow a $\mathbb{C}$-module $\underline{A}$ with an algebra structure: (ρ_A, μ_A, η_A) is an algebra object in $[U\mathfrak{g}, \mathscr{M}]$.

The collection of commutative algebra objects in $[U\mathfrak{g}, \mathscr{M}]$ together with $[U\mathfrak{g}, \mathscr{M}]$-morphisms $f : \rho_A \Rightarrow \rho_B$ which preserve the product μ_A and unit η_A, i.e.

$$f \circ \mu_A = \mu_B \circ (f \otimes_{\rho_{\mathbb{C}}} f) \ , \ f \circ \eta_A = \eta_B \circ \mathrm{id}_{\rho_{\mathbb{C}}} \ , \tag{4.1.7}$$

constitute a subcategory of $[U\mathfrak{g}, \mathscr{M}]$. This subcategory of commutative algebra objects is equivalent to the pair of comma categories $(\otimes_{\rho_{\mathbb{C}}} \Rightarrow \mathrm{id}_{[U\mathfrak{g}, \mathscr{M}]})$ and $(\mathrm{id}_{[U\mathfrak{g}, \mathscr{M}]} \Rightarrow \mathrm{id}_{[U\mathfrak{g}, \mathscr{M}]})$ whose objects are pairs of triples $(\rho_A \times \rho_A, \mu_A, \rho_A)$ and $(\rho_{\mathbb{C}}, \eta_A, \rho_A)$ with (ρ_A, μ_A, η_A) is a commutative monoid object in $[U\mathfrak{g}, \mathscr{M}]$, and whose morphisms are pairs of tuples of morphisms $(f \times f, f)$ and $(\mathrm{id}_{\rho_{\mathbb{C}}}, f)$ satisfying (4.1.7) (see Definition A.2.11 for the definition of a comma category). We shall denote by

$$U\mathfrak{g}\text{-}\mathrm{Alg}^{\mathrm{com}} \ , \tag{4.1.8}$$

the category of commutative algebras in $[U\mathfrak{g}, \mathscr{M}]$. And with an abuse of notation denote objects in $U\mathfrak{g}$-$\mathrm{Alg}^{\mathrm{com}}$ by the corresponding objects in $[U\mathfrak{g}, \mathscr{M}]$.

We now construct a functor

$$C^\infty : [G, \mathsf{Man}]^{\mathrm{op}} \longrightarrow U\mathfrak{g}\text{-}\mathrm{Alg}^{\mathrm{com}} \ . \tag{4.1.9}$$

For any object ρ_M in $[G, \mathsf{Man}]$ we set $C^\infty(\rho_M) := \rho_{C^\infty(M)}$. We denote by $\rho_{C^\infty(M)}(*) = C^\infty(\underline{M})$ the $\mathbb{C}$-vector space of smooth complex-valued functions on $\underline{M}$. The left $U\mathfrak{g}$-

action is induced by the G-action as

$$\rho_{C^\infty(M)}(\xi)(a) := \xi \triangleright_{C^\infty(M)} a := \frac{\mathrm{d}}{\mathrm{d}t}\big(a \circ \rho_M(\exp(-t\,\xi))\big)\Big|_{t=0} \ , \tag{4.1.10}$$

for all $\xi \in \mathfrak{g}$ and $a \in C^\infty(\underline{M})$. By $\exp : \mathfrak{g} \to G$ we denote the exponential map of the Lie group G. The product $\mu_{C^\infty(M)} : C^\infty(\rho_M) \otimes C^\infty(\rho_M) \Rightarrow C^\infty(\rho_M)$ has single component the usual pointwise multiplication of functions and the unit $\eta_{C^\infty(M)}$: $\rho_{\mathbb{C}} \Rightarrow C^\infty(\rho_M)$ has single component $c \mapsto c\,1_{C^\infty(M)}$ (the constant functions).

Using (4.1.10) and (4.1.1), it is easy to check that $\mu_{C^\infty(M)}$ and $\eta_{C^\infty(M)}$ are $[U\mathfrak{g}, \mathscr{M}]$-morphisms: As a consequence of the Leibniz or product rule for differentiation

$$\begin{aligned} \xi \triangleright_{C^\infty(M)} (a\,a') &= (\xi \triangleright_{C^\infty(M)} a)\,a' + a\,(\xi \triangleright_{C^\infty(M)} a') \\ &= (\xi_{(1)} \triangleright_{C^\infty(M)} a)\,(\xi_{(2)} \triangleright_{C^\infty(M)} a') \ , \end{aligned} \tag{4.1.11}$$

for all $a,\ a' \in C^\infty(M)$ and as a consequence of derivatives on constant functions being zero

$$\xi \triangleright_{C^\infty(M)} 1_{C^\infty(M)} = 0 = \epsilon(\xi)\,1_{C^\infty(M)} \ . \tag{4.1.12}$$

Hence $\rho_{C^\infty(M)}$ is an object in $U\mathfrak{g}$-Alg. For any morphism $f^{\mathrm{op}} : \rho_M \Rightarrow \rho_N$ in $[G, \mathsf{Man}]^{\mathrm{op}}$ (i.e. a smooth G-equivariant map $f : \underline{N} \to \underline{M}$) we set

$$C^\infty(f^{\mathrm{op}}) := f^* : C^\infty(\rho_M) \Longrightarrow C^\infty(\rho_N) \ , \tag{4.1.13}$$

with single component

$$f^* : C^\infty(\underline{M}) \Longrightarrow C^\infty(\underline{N}) \ , \quad a \longmapsto a \circ f \tag{4.1.14}$$

to be the pull-back of functions along f. Since f is G-equivariant, i.e

$$f \circ \rho_N(g) = \rho_M(g) \circ f \ , \tag{4.1.15}$$

it follows that $f^* = C^\infty(f^{\rm op})$ is $U\mathfrak{g}$-equivariant:

$$
\begin{aligned}
\xi \triangleright_{C^\infty(N)} \big(f^*(a)\big) &= \xi \triangleright_{C^\infty(N)} (a \circ f) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big((a\circ f)\circ \rho_N(\exp(-t\,\xi))\big)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(a\circ \rho_M(\exp(-t\,\xi))\big)\Big|_{t=0} \circ f \\
&= (\xi \triangleright_{C^\infty(M)} a)\circ f \\
&= f^*(\xi \triangleright_{C^\infty(M)} a) \ .
\end{aligned} \tag{4.1.16}
$$

Since pull-backs also preserve the products and units ($f^*(a\,a') = a\,a'\circ f = (a\circ f)\,(a'\circ f) = f^*(a)\,f^*(a')$ and $f^*(1_{C^\infty(M)}) = 1_{C^\infty(M)}\circ f = 1_{C^\infty(M)}$) and clearly is commutative, we have that $C^\infty(f^{\rm op}) : C^\infty(\rho_M) \Rightarrow C^\infty(\rho_N)$ is a morphism in $U\mathfrak{g}\text{-}\mathrm{Alg}^{\rm com}$. In summary, we have shown

Proposition 4.1.1. *There exists a functor $C^\infty : [G,\mathsf{Man}]^{\rm op} \to U\mathfrak{g}\text{-}\mathrm{Alg}$. Taking into account the triangular structure $R = 1\otimes 1$ on $U\mathfrak{g}$, the functor C^∞ is valued in the full subcategory $U\mathfrak{g}\text{-}\mathrm{Alg}^{\rm com}$ of braided commutative algebras in $[U\mathfrak{g},\mathscr{M}]$.*

Fixing any object ρ_M in $[G,\mathsf{Man}]$, we can consider the slice category (see A.2.12)

$$
G\text{-}\mathsf{VecBun}_M := \big([G,\mathsf{Man}] \Rightarrow \rho_M\big) \ , \tag{4.1.17}
$$

together with the condition that

$$
\rho_E(g) : \underline{E}_{\,x} \longrightarrow \underline{E}_{\,\rho_M(g)\,(x)} \ , \tag{4.1.18}
$$

is a $\mathbb{C}$-linear map, for any $(\rho_E,\pi_E)\in G\text{-}\mathsf{VecBun}_M$ and for all $g\in G$ and $x\in \underline{M}$. This is the category of G-equivariant vector bundles over M. The objects in $G\text{-}\mathsf{VecBun}_M$ are then pairs (ρ_E,π_E) consisting of a finite-rank complex vector bundle $\underline{E} \xrightarrow{\pi_E} \underline{M}$

over $\underline{M}$. The G-equivariance of $\pi_E : \rho_E \Rightarrow \rho_M$ is that the diagram

$$\begin{tikzcd} \underline{E} \arrow[r, "\rho_E(g)"] \arrow[d, "\pi_E"'] & \underline{E} \arrow[d, "\pi_E"] \\ \underline{M} \arrow[r, "\rho_M(g)"'] & \underline{M} \end{tikzcd} \tag{4.1.19}$$

commutes for any $g \in G$ (cf. (4.1.4)). A morphism in G-VecBun_M is a $[G, \mathsf{Man}]$-morphism

$$f : \rho_E \Longrightarrow \rho_{E'} \ , \tag{4.1.20}$$

with single component $f : \underline{E} \to \underline{E}'$ (denoted by the same symbol) such that the diagram

$$\begin{tikzcd} \underline{E} \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & \underline{E}' \arrow[dl, "\pi_{E'}"] \\ & \underline{M} & \end{tikzcd} \tag{4.1.21}$$

commutes in Man. The naturality condition of f is that

$$\begin{tikzcd} \underline{E} \arrow[r, "\rho_E(g)"] \arrow[d, "f"'] & \underline{E} \arrow[d, "f"] \\ \underline{E}' \arrow[r, "\rho_{E'}(g)"'] & \underline{E}' \end{tikzcd} \tag{4.1.22}$$

commutes for any $g \in G$. In other words morphisms in G-VecBun_M are G-equivariant vector bundle maps covering the identity $\mathrm{id}_{\underline{M}}$.

Remark 4.1.2. Combining the conditions (4.1.21) and (4.1.22) in a single commutative diagram

$$\begin{tikzcd}
\underline{E} \arrow[rr, "\rho_E(g)"] \arrow[dr, "\pi_E"] \arrow[dd, "f"'] & & \underline{E} \arrow[dl, "\pi_E"'] \arrow[dd, "f"] \\
& \underline{M} & \\
\underline{E}' \arrow[rr, "\rho_{E'}(g)"'] \arrow[ur, "\pi_{E'}"] & & \underline{E}' \arrow[ul, "\pi_{E'}"']
\end{tikzcd} \tag{4.1.23}$$

gives the additional condition that (cf. (4.1.18))

$$\rho_E(g) : \underline{E}_{\,x} \longrightarrow \underline{E}_{\,\rho_M(g)\,(x)} \stackrel{!}{=} \underline{E}_{\,x} \ . \tag{4.1.24}$$

(from the top and bottom triangles). Forgetting the linearity of the maps, this would force the bundle to be a principle bundle.

We now review the definition of the category of symmetric bimodules over an algebra object in $[U\mathfrak{g}, \mathscr{M}]$: Choosing one particular algebra $\rho_A \in U\mathfrak{g}\text{-Alg}^{\mathrm{com}}$ there is a morphism (left ρ_A-action) $l_V : \rho_A \otimes_{\rho_\mathbb{C}} \rho_V \Rightarrow \rho_V$ which endows a $\mathbb{C}$-module $\underline{V}$ with a left $\underline{A}$-module structure. The morphism $r_V := l_V \circ \tau_{V,A}$, where $\tau_{V,A}$ is the braiding morphism in the braided monoidal category $[U\mathfrak{g}, \mathscr{M}]$, is a right action of $\underline{A}$ on the $\mathbb{C}$-module $\underline{V}$. (ρ_V, l_V, r_V) satisfies the axioms of a symmetric bimodule object in $[U\mathfrak{g}, \mathscr{M}]$.

The collection of symmetric bimodule objects in $[U\mathfrak{g}, \mathscr{M}]$ together with $[U\mathfrak{g}, \mathscr{M}]$-morphisms $f : \rho_V \Rightarrow \rho_W$ which preserve the left ρ_A-action, i.e. such that

$$l_W \circ (\mathrm{id}_{\rho_A} \otimes_{\rho_\mathbb{C}} f) = f \circ l_V \ , \tag{4.1.25}$$

(the right ρ_A-action is automatically preserved) constitute a subcategory of $[U\mathfrak{g}, \mathscr{M}]$. This subcategory of bimodule objects is equal to the comma category $(\otimes_{\rho_\mathbb{C}} \Rightarrow \mathrm{id}_{[U\mathfrak{g}, \mathscr{M}]})$ whose objects are triples $(\rho_A \times \rho_V, l_V, \rho_V)$ and whose morphisms are tuples of morphisms $(\mathrm{id}_{\rho_A} \times f, f)$ satisfying (4.1.25) with $(\rho_V, l_V, l_V \circ \tau_{V,A})$ is a symmetric bimodule object in $[U\mathfrak{g},\mathscr{M}]$. We shall denote by

$$U\mathfrak{g}\text{-Bimod}(C^\infty(M))^{\mathrm{sym}} \ , \tag{4.1.26}$$

the category of symmetric bimodules in $[U\mathfrak{g},\mathscr{M}]$. And with an abuse of notation denote objects in $U\mathfrak{g}$-Bimod$(C^\infty(M))^{\mathrm{sym}}$ by the corresponding objects in $[U\mathfrak{g},\mathscr{M}]$.

We shall now construct a functor $\Gamma^\infty : G\text{-}\mathsf{VecBun}_M \to U\mathfrak{g}\text{-Bimod}(C^\infty(M))$. For any object (ρ_E,π_E) in $G\text{-}\mathsf{VecBun}_M$ we set $\Gamma^\infty((\rho_E,\pi_E)) := \rho_{\Gamma^\infty(E)}$, where $\rho_{\Gamma^\infty(E)}(*) = \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$ is the $\mathbb{C}$-vector space of smooth sections of $\underline{E} \xrightarrow{\pi_E} \underline{M}$ and the left $U\mathfrak{g}$-action is induced by the G-actions as

$$\rho_{\Gamma^\infty(E)}(\xi)(s) := \xi \triangleright_{\Gamma^\infty(E)} s := \frac{\mathrm{d}}{\mathrm{d}t}\big(\rho_E(\exp(t\,\xi),\,\cdot\,)\circ s\circ\rho_M(\exp(-t\,\xi),\,\cdot\,)\big)\Big|_{t=0} \ , \tag{4.1.27}$$

for all $\xi \in \mathfrak{g}$ and $s \in \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$. Notice that $\xi \triangleright_{\Gamma^\infty(E)} s$ is an element of $\Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$, i.e. it satisfies $\pi_E \circ (\xi \triangleright_{\Gamma^\infty(E)} s) = \mathrm{id}_{\underline{M}}$, since

$$\begin{aligned}
\pi_E\circ\rho_E(g,\,\cdot\,)\circ s\circ\rho_M(g^{-1},\,\cdot\,) &= \rho_M(g,\,\cdot\,)\circ\pi_E\circ s\circ\rho_M(g^{-1},\,\cdot\,)\\
&= \rho_M(g,\,\cdot\,)\circ\mathrm{id}_{\underline{M}}\circ\rho_M(g^{-1},\,\cdot\,)\\
&= \mathrm{id}_{\underline{M}} \ ,
\end{aligned} \tag{4.1.28}$$

for all $s \in \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$ and $g \in G$. In the first step we used the G-equivariance condition (4.1.19) and in the second step the fact that s is a section. The left and right $\rho_{C^\infty(M)}$-actions $l_{\Gamma^\infty(E)} : C^\infty(\rho_M) \otimes_{\rho_{\mathbb{C}}} \Gamma^\infty(\rho_E,\pi_E) \Rightarrow \Gamma^\infty(\rho_E,\pi_E)$ and $r_{\Gamma^\infty(E)} : \Gamma^\infty(\rho_E,\pi_E) \otimes_{\rho_{\mathbb{C}}} C^\infty(\rho_M) \Rightarrow \Gamma^\infty(\rho_E,\pi_E)$ have single component defined as usual pointwise. Using (4.1.10) and (4.1.27) it is easy to check that $l_{\Gamma^\infty(E)}$ and $r_{\Gamma^\infty(E)}$ are

$[U\mathfrak{g}, \mathscr{M}]$-morphisms:

$$
\begin{aligned}
\xi \triangleright_{\Gamma^\infty(E)} (a\, s) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\rho_E(g(t),\,\cdot\,)\circ(a\, s)\circ\rho_M(g(-t),\,\cdot\,)\big)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho_E(g(t),\,\cdot\,)\circ\big(\big(a\circ\rho_M(g(-t),\,\cdot\,)\big)\,\cdot\,\big(s\circ\rho_M(g(-t),\,\cdot\,)\big)\big)\Big)\Big|_{t=0} \\
&= (\xi\triangleright_{C^\infty(M)} a)\, s + a\,(\xi\triangleright_{\Gamma^\infty(E)} s)\ ,
\end{aligned} \tag{4.1.29}
$$

for any $a \in C^\infty(\underline{M}), s \in \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$ where the second equality follows from the definition of the pointwise product of functions and the last equality from the product rule for differentiation (the right action follows automatically). It is also simple to check that $\Gamma^\infty(\rho_E, \pi_E)$ is an object in $U\mathfrak{g}$-Bimod$(C^\infty(M))^{\mathrm{sym}}$. For this we need to show that the bimodule axioms hold. This is a simple consequence of the associativity of the pointwise multiplication of functions. For any morphism $f : \rho_E \Rightarrow \rho_{E'}$ in G-VecBun_M we set

$$
\Gamma^\infty(f) : \Gamma^\infty(\rho_E, \pi_E) \longrightarrow \Gamma^\infty(\rho_{E'}, \pi_{E'}\,)\ , \tag{4.1.30}
$$

to have single component given by $s \longmapsto f\circ s$. By the commutative diagram (4.1.21) it follows that $\Gamma^\infty(f)(s)$ is a section of $\underline{E'} \xrightarrow{\pi_{E'}} \underline{M}$ (since $\pi_{E'}\circ\Gamma^\infty(f)(s) = \pi_{E'}\circ f\circ s = \pi_E\circ s = \mathrm{id}_M$ for all $s \in \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$) and the diagram (4.1.22) implies that $\Gamma^\infty(f)$ is $U\mathfrak{g}$-equivariant (since $\Gamma^\infty(f)(\rho_E(g)) = f\circ\rho_E(g) = \rho_{E'}(g)\circ f = \rho_{E'}(g)\circ\Gamma^\infty(f)$). One easily checks that $\Gamma^\infty(f)$ preserves the left and right $C^\infty(\rho_M)$-module structures (by the left $C^\infty(\underline{M})$-linearity of f, $\Gamma^\infty(f)(a\, s) = f(a\, s) = a\, f(s) = a\,\Gamma^\infty(f)(s)$ for all $a \in C^\infty(\underline{M})$ and $s \in \Gamma^\infty(\underline{E} \xrightarrow{\pi_E} \underline{M})$, and similarly for the right $C^\infty(\underline{M})$-action) and that these are commutative. Hence we find that $\Gamma^\infty(f) : \Gamma^\infty(\rho_E, \pi_E) \to \Gamma^\infty(\rho_{E'}, \pi_{E'}\,)$ is a morphism in $U\mathfrak{g}$–Bimod$(C^\infty(M))^{\mathrm{sym}}$. In summary, we have shown

Proposition 4.1.3. *For any G-manifold $\rho_M \in [G, \mathsf{Man}]$ with $\mathfrak{g}$ the Lie algebra of G there exists a functor $\Gamma^\infty : G$-$\mathsf{VecBun}_M \to U\mathfrak{g}$–Bimod$(C^\infty(M))^{\mathrm{sym}}$.*

The functor Γ^∞ in Proposition 4.1.3 is in fact a braided closed monoidal functor with respect to the braided closed monoidal structure on G-VecBun_M that we shall now describe. Firstly, notice that G-VecBun_M is a monoidal category: The (fibrewise) tensor product $\underline{E} \otimes \underline{E'}$ of two G-equivariant vector bundles $\underline{E}, \underline{E'}$ is again a G-equivariant vector bundle with respect to the diagonal left G-action $\triangleright_{E\otimes E'} : G \times (\underline{E} \otimes \underline{E'}\,) \to \underline{E} \otimes \underline{E'}\,,\ (g, e \otimes e'\,) \mapsto \rho_E(g, e) \otimes \rho_{E'}(g, e'\,)$. We therefore have a functor $\otimes : G\text{-}\mathsf{VecBun}_M \times G\text{-}\mathsf{VecBun}_M \to G\text{-}\mathsf{VecBun}_M$. The trivial line bundle $\mathbb{C} \times \underline{M}$ (with trivial G-action on the fibres) is the unit object $\rho_{\mathbb{C}} \times \rho_M$ in $G\text{-}\mathsf{VecBun}_M$, the components of the associator are the identities and the unitors are the obvious ones. Hence $G\text{-}\mathsf{VecBun}_M$ is a monoidal category. The (fibrewise) flip map $\tau_{E,E'} : \underline{E}\otimes\underline{E'} \to \underline{E'}\otimes\underline{E}$ turns $G\text{-}\mathsf{VecBun}_M$ into a braided (and even symmetric) monoidal category. Secondly, notice that $G\text{-}\mathsf{VecBun}_M$ has an internal hom-functor which turns it into a braided closed monoidal category: For any two G-equivariant vector bundles $\rho_E, \rho_{E'}$ we can form the homomorphism bundle $\hom(\rho_E, \rho_{E'}\,)$ which is a G-equivariant vector bundle with respect to the left adjoint G-action $\triangleright_{\hom(E,E'\,)} : G \times \hom(\underline{E}, \underline{E'}\,) \to \hom(\underline{E}, \underline{E'}\,)\,,\ (g, L) \mapsto \rho_{E'}(g) \circ L \circ \rho_E(g^{-1})$. The currying maps $\zeta_{E,E',E''} : \mathrm{Hom}_{G\text{-}\mathsf{VecBun}_M}(\rho_E \otimes \rho_{E'}, \rho_{E''}\,) \Rightarrow \mathrm{Hom}_{G\text{-}\mathsf{VecBun}_M}(\rho_E, \hom(\rho_{E'}, \rho_{E''}\,))$ are given by assigning to any $G\text{-}\mathsf{VecBun}_M$-morphism $f : \rho_E \otimes \rho_{E'} \Rightarrow \rho_{E''}$ the $G\text{-}\mathsf{VecBun}_M$-morphism

$$\zeta_{E,E',E''}(f) : \rho_E \Longrightarrow \hom(\rho_{E'}, \rho_{E''}\,)\ , \tag{4.1.31}$$

with single component $e \mapsto f(e \otimes \,\cdot\,)$. Making use now of the standard natural isomorphisms

$$\Gamma^\infty(\underline{E} \otimes \underline{E'}\,) \simeq \Gamma^\infty(\underline{E}) \otimes_{C^\infty(M)} \Gamma^\infty(\underline{E'}\,)\ , \tag{4.1.32a}$$

$$\Gamma^\infty(\mathbb{C} \times \underline{M}) \simeq C^\infty(\underline{M})\ , \tag{4.1.32b}$$

$$\Gamma^\infty(\hom(\underline{E}, \underline{E'}\,)) \simeq \hom_{C^\infty(M)}(\Gamma^\infty(\underline{E}), \Gamma^\infty(\underline{E'}\,))\ , \tag{4.1.32c}$$

we obtain

Proposition 4.1.4. *The functor $\Gamma^\infty : G\text{-}\mathsf{VecBun}_M \to U\mathfrak{g}\text{–}\mathrm{Bimod}(C^\infty(M))^{\mathrm{sym}}$ of Proposition 4.1.3 is a braided closed monoidal functor.*

Before we can deform the categories $U\mathfrak{g}\text{–}\mathrm{Alg}^{\mathrm{com}}$ and $U\mathfrak{g}\text{–}\mathrm{Bimod}(C^\infty(M))^{\mathrm{sym}}$ via cochain twists F, we have to introduce formal power series extensions in a deformation parameter $\hbar$ of all $\mathbb{C}$-vector spaces involved, which then become $\mathbb{C}[[\hbar]]$-modules. For details on formal power series and the $\hbar$-adic topology see [43, Chapter XVI]. We shall denote the $\hbar$-adic topological tensor product by $\widehat{\otimes}$ and recall that it satisfies $\underline{V}[[\hbar]] \,\widehat{\otimes}\, \underline{W}[[\hbar]] \simeq (\underline{V} \otimes \underline{W})[[\hbar]]$, where $\underline{V}, \underline{W}$ are $\mathbb{C}$-vector spaces and $\underline{V}[[\hbar]], \underline{W}[[\hbar]]$ are the corresponding topologically free $\mathbb{C}[[\hbar]]$-modules. Let us denote by $U\mathfrak{g}[[\hbar]]$ the formal power series extension of the cocommutative Hopf algebra $U\mathfrak{g}$ (the product and coproduct here involves the topological tensor product $\widehat{\otimes}$) and by $[U\mathfrak{g}[[\hbar]], \mathscr{M}]$ the braided closed monoidal category of left $U\mathfrak{g}[[\hbar]]$-modules over $\mathbb{C}[[\hbar]]$ (with monoidal structure given by $\widehat{\otimes}$). There is a braided closed monoidal functor $[[\hbar]] : [U\mathfrak{g}, \mathscr{M}] \to [U\mathfrak{g}[[\hbar]], \mathscr{M}]$: To any object ρ_V in $[U\mathfrak{g}, \mathscr{M}]$ we assign the object $\rho_{V[[\hbar]]}$ in $[U\mathfrak{g}[[\hbar]], \mathscr{M}]$ and to any $[U\mathfrak{g}, \mathscr{M}]$-morphism $f : \rho_V \Rightarrow \rho_W$ we assign the $[U\mathfrak{g}[[\hbar]], \mathscr{M}]$-morphism $f : \rho_{V[[\hbar]]} \Rightarrow \rho_{W[[\hbar]]}$ with single component (denoted by the same symbol)

$$f : \underline{V}[[\hbar]] \longrightarrow \underline{W}[[\hbar]] \ , \quad v = \sum_{n=0}^{\infty} \hbar^n \, v_n \longmapsto f(v) = \sum_{n=0}^{\infty} \hbar^n \, f(v_n) \ . \tag{4.1.33}$$

The functor $[[\hbar]]$ is a braided closed monoidal functor due to the natural isomorphisms

$$\underline{V}[[\hbar]] \,\widehat{\otimes}\, \underline{W}[[\hbar]] \simeq (\underline{V} \otimes \underline{W})[[\hbar]] \ , \tag{4.1.34a}$$

$$\hom_{[[\hbar]]}(\underline{V}[[\hbar]], \underline{W}[[\hbar]]) \simeq \hom(\underline{V}, \underline{W})[[\hbar]] \ . \tag{4.1.34b}$$

Here we have denoted by $\hom_{[[\hbar]]}$ the internal hom-functor in $[U\mathfrak{g}[[\hbar]], \mathscr{M}]$. As a consequence of (4.1.34a) this functor induces a functor $[[\hbar]] : U\mathfrak{g}\text{–}\mathrm{Alg}^{\mathrm{com}} \to U\mathfrak{g}[[\hbar]]\text{–}\mathrm{Alg}^{\mathrm{com}}$ and a braided closed monoidal functor $[[\hbar]] : U\mathfrak{g}\text{–}\mathrm{Bimod}(C^\infty(M))^{\mathrm{sym}} \to U\mathfrak{g}[[\hbar]]\text{–}\mathrm{Bimod}(C^\infty(M)[[\hbar]])^{\mathrm{sym}}$.

Given now any cochain twist $F \in U\mathfrak{g}[[\hbar]] \,\widehat{\otimes}\, U\mathfrak{g}[[\hbar]]$ based on $U\mathfrak{g}[[\hbar]]$, Proposition 2.2.25 implies that there is a functor

$$\mathcal{F} : U\mathfrak{g}[[\hbar]]\text{–}\mathrm{Alg}^{\mathrm{com}} \longrightarrow U\mathfrak{g}[[\hbar]]_F\text{–}\mathrm{Alg}^{\mathrm{com}} \tag{4.1.35a}$$

and Proposition 2.3.5 and Theorem 2.3.19 imply that there is a braided closed monoidal functor

$$\mathcal{F} : U\mathfrak{g}[[\hbar]]\text{–Bimod}(C^\infty(M)[[\hbar]])^{\mathrm{sym}} \longrightarrow U\mathfrak{g}[[\hbar]]_F\text{–Bimod}(C^\infty(M)[[\hbar]]_F)^{\mathrm{sym}} \ . \tag{4.1.35b}$$

Precomposing these functors with the functors of Propositions 4.1.1 and 4.1.3 together with $[[\hbar]]$ yields the main result of this section.

Corollary 4.1.5. *Given any cochain twist $F \in U\mathfrak{g}[[\hbar]] \,\widehat{\otimes}\, U\mathfrak{g}[[\hbar]]$ there is the functor*

$$\begin{tikzcd} G\text{-}\mathsf{Man}^{\mathrm{op}} \arrow[r, "C^\infty_F"] \arrow[d, "C^\infty"'] & U\mathfrak{g}[[\hbar]]_F\text{–Alg}^{\mathrm{com}} \\ U\mathfrak{g}\text{–Alg}^{\mathrm{com}} \arrow[r, "{[[\hbar]]}"'] & U\mathfrak{g}[[\hbar]]\text{–Alg}^{\mathrm{com}} \arrow[u, "\mathcal{F}"'] \end{tikzcd} \tag{4.1.36a}$$

and the braided closed monoidal functor

$$\begin{tikzcd} G\text{-}\mathsf{VecBun}_M \arrow[r, "\Gamma^\infty_F"] \arrow[d, "\Gamma^\infty"'] & U\mathfrak{g}[[\hbar]]_F\text{–Bimod}(C^\infty(M)[[\hbar]]_F)^{\mathrm{sym}} \\ U\mathfrak{g}\text{–Bimod}(C^\infty(M))^{\mathrm{sym}} \arrow[r, "{[[\hbar]]}"'] & U\mathfrak{g}[[\hbar]]\text{–Bimod}(C^\infty(M)[[\hbar]])^{\mathrm{sym}} \arrow[u, "\mathcal{F}"'] \end{tikzcd} \tag{4.1.36b}$$

describing the formal deformation quantization of G-manifolds and G-equivariant vector bundles.

The functors in Corollary 4.1.5 enable us to make the following definitions in the usual spirit of noncommutative geometry:

Definition 4.1.6 (Nonassociative space)**.** By nonassociative space we mean a noncommutative and nonassociative algebra.

Definition 4.1.7 (Nonassociative vector bundle)**.** By nonassociative vector bundle we mean a noncommutative and nonassociative bimodule over a noncommutative and nonassociative algebra.

Algebra objects ρ_A in H–$\mathrm{Alg}^{\mathrm{com}}$ are to be interpreted as noncommutative and nonassociative spaces with symmetries modelled on the quasi-Hopf algebra H, while the ρ_A-bimodules ρ_V in H–Bimod$(A)^{\mathrm{sym}}$ are to be thought of as noncommutative and nonassociative vector bundles over ρ_A.

4.1.1 Examples

As we have seen above, classical examples of braided commutative algebras are given by function algebras $C^\infty(\underline{M})$ on G-manifolds $\underline{M}$, where G is a Lie group with Lie algebra $\mathfrak{g}$, the relevant triangular quasi-Hopf algebra in this case being the universal enveloping Hopf algebra $U\mathfrak{g}$ (with trivial R-matrix and associator). These examples and cochain twist deformations thereof are our main examples of interest.

Example 4.1.8. Let $G = \mathbb{T}^n$ be the n-dimensional torus, with $n \in \mathbb{N}$. Taking a basis $\{t_i \in \mathfrak{g} : i = 1, \dots, n\}$ of the Abelian Lie algebra $\mathfrak{g}$ and a skew-symmetric real-valued $n \times n$-matrix $\Theta = \big(\Theta^{ij}\big)_{i,j=1}^n$, we have the Abelian twist

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\Theta^{ij}\,t_i \otimes t_j\big) \tag{4.1.37}$$

based on $U\mathfrak{g}[[\hbar]]$ (with implicit sums over repeated upper and lower indices). The twisted Hopf algebra $U\mathfrak{g}[[\hbar]]_F$ is cocommutative (in fact $\Delta_F = \Delta$), and since F is a cocycle twist the algebras and bimodules obtained from the functors in Corollary 4.1.5 are strictly associative; however in general they are not strictly commutative as the twisted triangular structure is given by $R_F = F^{-2}$. This is the triangular Hopf algebra relevant to the standard noncommutative tori, and more generally to the toric noncommutative manifolds (or isospectral deformations) in the sense of [24].

Example 4.1.9. Fix $n \in \mathbb{N}$ and let $\mathfrak{g}$ be the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \le i < j \le n\}$ and Lie bracket relations given by

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j\ , \tag{4.1.38}$$

and all other Lie brackets equal to zero. Let us denote by G the Lie group obtained by Lie-integration of $\mathfrak{g}$ and notice that G is a Lie subgroup of $\mathrm{ISO}(2n)$. We fix a rank-three skew-symmetric real-valued tensor $\mathsf{R} = \big(\mathsf{R}^{ijk}\big)_{i,j,k=1}^{n}$ and introduce the non-Abelian cochain twist (with implicit summation over repeated upper and lower indices)

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac{1}{4}\,\mathsf{R}^{ijk}\,(m_{ij}\otimes t_k - t_i\otimes m_{jk}) + t_i\otimes\tilde{t}^{\,i} - \tilde{t}^{\,i}\otimes t_i\big)\big) \ . \tag{4.1.39}$$

The twisted quasi-Hopf algebra $U\mathfrak{g}[[\hbar]]_F$ is non-cocommutative: the twisted coproduct on primitive elements is given by

$$\Delta_F(t_i) = \Delta(t_i) \ , \tag{4.1.40a}$$

$$\Delta_F(\tilde{t}^{\,i}) = \Delta(\tilde{t}^{\,i}) + \tfrac{\mathrm{i}\,\hbar}{2}\,\mathsf{R}^{ijk}\,t_j\otimes t_k \ , \tag{4.1.40b}$$

$$\Delta_F(m_{ij}) = \Delta(m_{ij}) - \mathrm{i}\,\hbar\,(t_i\otimes t_j - t_j\otimes t_i) \ . \tag{4.1.40c}$$

Generally the algebras and bimodules obtained from the functors in Corollary 4.1.5 are noncommutative and nonassociative: the twisted triangular structure is given by $R_F = F^{-2}$, while a straightforward calculation of (2.1.109) with $\phi = 1\otimes 1\otimes 1$ using the Baker-Campbell-Hausdorff formula yields the associator

$$\phi_F = \exp\big(\tfrac{\hbar^2}{2}\,\mathsf{R}^{ijk}\,t_i\otimes t_j\otimes t_k\big) \ . \tag{4.1.41}$$

This is the triangular quasi-Hopf algebra relevant in the phase space formulation for the nonassociative deformations of geometry that arise in non-geometric R-flux backgrounds of string theory [27].

4.2 Nonassociative spaces and vector bundles

In this section we review aspects of our formalism for differential geometry on noncommutative and nonassociative spaces which arise from cochain twist deformation quantization of manifolds by working in the simplest setting of trivial vector bundles. We use mainly the infix action notation rather than representation notation as is conventional in the physics literature.

4.2.1 Spaces

Let M be a manifold in Man. In the following we fix a choice of sub-Hopf algebra $H \subseteq U\mathrm{Vec}(M)$, which we shall interpret as the symmetries of M along which we want to perform the deformation quantization. See Examples 4.2.1 and 4.2.2 below for typical choices in the context of flux compactifications of closed string theory.

Let us denote by $A := C^\infty(M)$ the algebra of complex-valued smooth functions on M. The action of vector fields on A as derivations can be extended to an H-action $\triangleright : H \otimes A \to A$, which preserves the product and unit in A, i.e.

$$h \triangleright (a\, b) = \big(h_{(1)} \triangleright a\big)\, \big(h_{(2)} \triangleright b\big) \ , \quad h \triangleright 1 = \epsilon(h)\, 1 \ , \tag{4.2.1}$$

for all $h \in H$ and $a, b \in A$. Here we have used the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (with summations understood) to abbreviate the coproduct. In technical terms (4.2.1) states that A is an H-module algebra.

The commutative and associative algebra A can be deformed by using a cochain twist F of H into a noncommutative cochain twist F and nonassociative algebra $A_\star$. The product μ in the algebra A is deformed using the cochain twist F to a noncommutative and nonassociative $\star$-product

$$\mu_\star := \mu \circ F^{-1} \ . \tag{4.2.2}$$

We denote the resulting noncommutative and nonassociative algebra by $A_\star$ and abbreviate the $\star$-product as $a \star b := \mu_\star(a \otimes b)$, for $a, b \in A_\star$. In the spirit of noncommutative geometry, we interpret the algebra $A_\star$ as (the algebra of functions on) a noncommutative and nonassociative space.

By construction, the original H-action $\triangleright : H \otimes A \to A$ induces an H_F-action $\triangleright : H_F \otimes A_\star \to A_\star$, which preserves the product and unit in $A_\star$, i.e.

$$h \triangleright (a \star b) = \big(h_{(1)_F} \triangleright a\big) \star \big(h_{(2)_F} \triangleright b\big) \ , \quad h \triangleright 1 = \epsilon_F(h)\, 1 \ , \tag{4.2.3}$$

for all $h \in H_F$ and $a, b \in A_\star$. Indeed we have by the definition of the twisted coproduct Δ_F in H_F that

$$
\begin{aligned}
h \triangleright (a \star b) &= \mu \circ (\rho_A \otimes \rho_A)(\Delta(h)\, F^{-1})(a \otimes_F b) \\
&= \mu \circ (\rho_A \otimes \rho_A)(F^{-1}\, F\Delta(h)\, F^{-1})(a \otimes_F b) \\
&= \mu \circ (\rho_A \otimes \rho_A)(F^{-1}\, \Delta_F(h))(a \otimes_F b) \\
&= \mu_\star \circ (\rho_A \otimes \rho_A)(\Delta_F(h))(a \otimes_F b) \\
&= \big(h_{(1)_F} \triangleright a\big) \star \big(h_{(2)_F} \triangleright b\big) \ .
\end{aligned}
\tag{4.2.4}
$$

and by the definition $\epsilon_F = \epsilon$ that

$$
h \triangleright 1 = \epsilon(h)\, 1 = \epsilon_F(h)\, 1 \ , \tag{4.2.5}
$$

for any $h \in H_F$ and $a, b \in A_\star$. Here we have used the Sweedler notation $\Delta_F(h) = h_{(1)_F} \otimes h_{(2)_F}$ (with summations understood) to abbreviate the deformed coproduct. It is important to observe that the noncommutativity of $A_\star$ is controlled by the triangular R-matrix

$$
R_F = F_{21}\, R\, F^{-1} = R_F^{(1)} \otimes R_F^{(2)} \tag{4.2.6}
$$

in $H_F \otimes H_F$, where $R = 1 \otimes 1$ in the original Hopf algebra H and $F_{21} = F^{(2)} \otimes F^{(1)}$ is the twist with flipped legs. Explicitly, the $\star$-product is commutative up to the action of R_F, i.e.

$$
a \star b = \big(R_F^{(2)} \triangleright b\big) \star \big(R_F^{(1)} \triangleright a\big) \ , \tag{4.2.7}
$$

for all $a, b \in A_\star$. Similarly, the nonassociativity of $A_\star$ is controlled by the associator $\phi_F = \phi_F^{(1)} \otimes \phi_F^{(2)} \otimes \phi_F^{(3)}$ in $H_F \otimes H_F \otimes H_F$ given by (2.1.109) with $\phi = 1 \otimes 1 \otimes 1$ in the original Hopf algebra H. Explicitly, the $\star$-product is associative up to the action of

ϕ_F, i.e.

$$(a \star b) \star c = (\phi_F^{(1)} \triangleright a) \star \big((\phi_F^{(2)} \triangleright b) \star (\phi_F^{(3)} \triangleright c) \big) \ , \tag{4.2.8}$$

for all $a, b, c \in A_\star$. In technical terms (4.2.3) together with (4.2.2), (4.2.7) and (4.2.8) states that $A_\star$ is a braided commutative H_F-module algebra.

We revisit examples 4.1.8 and 4.1.9 in the context of flux compactifications of closed string theory.

Example 4.2.1 (Q-flux compactification)**.** Let $M = \mathbb{R}^m$ and consider the Abelian cocycle twist (with summation over $i, j, \dots$ understood here and in the following)

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\, \Theta^{ij}\, P_i \otimes P_j \big) \tag{4.2.9}$$

based on the cocommutative Hopf algebra $H = U\mathfrak{g}$, where $\mathfrak{g}$ is the Abelian Lie algebra of infinitesimal translations $\{P_i : 1 \le i \le m\}$ and $\Theta = (\Theta^{ij})_{i,j=1}^m = (\mathsf{Q}^{ij}{}_k\, w^k)_{i,j=1}^m$, with w^k coordinates on the phase space, is an antisymmetric real-valued $m \times m$-matrix which arises from a constant non-geometric Q-flux of closed string theory [20, 10]. In this example we have

$$R_F = F^{-2} = \exp\big(\mathrm{i}\,\hbar\, \Theta^{ij}\, P_i \otimes P_j \big) \quad , \qquad \phi_F = 1 \otimes 1 \otimes 1 \ . \tag{4.2.10}$$

In particular $A_\star$ is strictly associative for this choice of twist.

Example 4.2.2 (R-flux compactification)**.** Let $M = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and consider the non-Abelian cochain twist

$$F = \exp\Big(-\tfrac{\mathrm{i}\,\hbar}{2} \big(\tfrac{1}{4}\, \mathsf{R}^{ijk}\, (M_{ij} \otimes P_k - P_i \otimes M_{jk}) + P_i \otimes \tilde{P}^{\,i} - \tilde{P}^{\,i} \otimes P_i \big) \Big) \tag{4.2.11}$$

based on the cocommutative Hopf algebra $H = U\mathfrak{g}$, where $\mathfrak{g}$ is the non-Abelian nilpotent Lie algebra of infinitesimal translations and Bopp shifts $\{P_i, \tilde{P}^{\,i}, M_{ij} : 1 \le i < j \le n\}$; the nontrivial Lie bracket relations are given by $[\,\tilde{P}^{\,i}, M_{jk}] = \delta^i{}_j\, P_k - \delta^i{}_k\, P_j$. Here $\mathsf{R} = (\mathsf{R}^{ijk})_{i,j,k=1}^n$ is a completely antisymmetric real-valued tensor of rank 3 which arises from a constant non-geometric R-flux of closed string theory [27]. In this example we have

$$R_F = F^{-2} \quad , \qquad \phi_F = \exp\left(\tfrac{\hbar^2}{2}\, \mathsf{R}^{ijk}\, P_i \otimes P_j \otimes P_k\right) . \tag{4.2.12}$$

In particular $A_\star$ is *not* strictly associative for this choice of twist.

4.2.2 Vector bundles

Given any (complex) vector bundle $E \to M$ over the manifold M, we can consider its smooth sections $\Gamma^\infty(E)$, which is a bimodule over $A = C^\infty(M)$ with respect to the usual pointwise module structures. To simplify our considerations in this section, we assume that $E \to M$ is a trivial complex vector bundle of rank n, i.e. $E = M \times \mathbb{C}^n \to M$ with bundle projection given by projecting on the first factor.

The sections of a trivial vector bundle over M of rank n can be described by a free A-bimodule $V = A^n$. Elements $v \in V$ are thus given by column vectors with entries in A (cf. Examples 2.1.17 and 2.3.4), i.e.

$$v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \quad , \qquad v^i \in A \ , \quad i = 1, \dots, n \ . \tag{4.2.13}$$

Alternatively, we can make use of the standard basis $\{e_i\}_{i=1}^n$ and write

$$v = e_i\, v^i \quad , \qquad v^i \in A \ , \quad i = 1, \dots, n \ . \tag{4.2.14}$$

The left and right A-actions on V are given componentwise, i.e.

$$a\ v := e_i\, (a\ v^i) \ , \tag{4.2.15a}$$

$$v\ a := e_i\, (v^i\ a) \ , \tag{4.2.15b}$$

for all $a \in A$ and $v \in V$. Similarly, we equip V with a componentwise H-action

$\triangleright : H \otimes V \to V$, i.e.

$$h \triangleright v := e_i \, (h \triangleright v^i) \ , \tag{4.2.16}$$

for all $h \in H$ and $v \in V$. It follows that

$$h \triangleright e_i = \epsilon(h) \, e_i \ , \tag{4.2.17}$$

for all $h \in H$ and $i = 1, \ldots, n$, i.e. the basis $\{e_i\}_{i=1}^n$ is H-invariant. As a consequence of (4.2.1), we obtain further that

$$h \triangleright (a \ v) = \big(h_{(1)} \triangleright a\big) \ \big(h_{(2)} \triangleright v\big) \ , \tag{4.2.18a}$$

$$h \triangleright (v \ a) = \big(h_{(1)} \triangleright v\big) \ \big(h_{(2)} \triangleright a\big) \ , \tag{4.2.18b}$$

for all $a \in A$, $v \in V$ and $h \in H$. In technical terms (4.2.18) states that V is an H-module bimodule over the H-module algebra A.

We have explained how a twist $F \in H \otimes H$ can be used to deform the Hopf algebra H to a quasi-Hopf algebra H_F, and the commutative and associative algebra A to a noncommutative and nonassociative algebra $A_\star$. Similarly, we can deform V into an H_F-module $A_\star$-bimodule $V_\star$ by introducing the H_F and $A_\star$-actions

$$h \triangleright v := e_i \, (h \triangleright v^i) \ , \tag{4.2.19a}$$

$$a \star v := e_i \, (a \star v^i) \ , \tag{4.2.19b}$$

$$v \star a := e_i \, (v^i \star a) \ , \tag{4.2.19c}$$

for all $h \in H_F$, $a \in A_\star$ and $v \in V_\star$. One easily verifies the compatibility conditions between the H_F and $A_\star$-actions

$$h \triangleright (a \star v) = \big(h_{(1)_F} \triangleright a\big) \star \big(h_{(2)_F} \triangleright v\big) \ , \tag{4.2.20a}$$

$$h \triangleright (v \star a) = \big(h_{(1)_F} \triangleright v\big) \star \big(h_{(2)_F} \triangleright a\big) \ , \tag{4.2.20b}$$

for all $h \in H_F$, $a \in A_\star$ and $v \in V_\star$. In the spirit of noncommutative geometry, we interpret $V_\star$ as (the module of sections of) a vector bundle over $A_\star$.

Noncommutativity of the $A_\star$-bimodule structure is controlled as in (4.2.7) by the R-matrix R_F, i.e.

$$a \star v = \big(R_F^{(2)} \triangleright v\big) \star \big(R_F^{(1)} \triangleright a\big) \ , \tag{4.2.21a}$$

$$v \star a = \big(R_F^{(2)} \triangleright a\big) \star \big(R_F^{(1)} \triangleright v\big) \ , \tag{4.2.21b}$$

for all $a \in A_\star$ and $v \in V_\star$, while nonassociativity is controlled as in (4.2.8) by the associator ϕ_F, i.e.

$$(a \star b) \star v = (\phi_F^{(1)} \triangleright a) \star \big((\phi_F^{(2)} \triangleright b) \star (\phi_F^{(3)} \triangleright v)\big) \ , \tag{4.2.22a}$$

$$v \star (a \star b) = \big((\phi_F^{(-1)} \triangleright v) \star (\phi_F^{(-2)} \triangleright a)\big) \star (\phi_F^{(-3)} \triangleright b) \ , \tag{4.2.22b}$$

for all $a, b \in A_\star$ and $v \in V_\star$. Here we have denoted the components of the inverse associator by $\phi_F^{-1} = \phi_F^{(-1)} \otimes \phi_F^{(-2)} \otimes \phi_F^{(-3)}$ (with summations understood).

4.2.3 Homomorphism bundles

Many interesting objects in differential geometry are described by maps between vector bundles. For example, the curvature of a connection on a vector bundle $E \to M$ is a map $E \to E \otimes \bigwedge^2 T^*M$ where $\bigwedge^2 T^*M$ is the exterior bundle of alternating 2-forms on the tangent bundle. Recall that vector bundle maps between two vector bundles $E \to M$ and $E' \to M$ can be equivalently described by sections of the homomorphism bundle $\mathrm{hom}(E, E') \to M$ (cf. (4.1.21)). The module of sections $\Gamma^\infty(\mathrm{hom}(E, E'))$ of the homomorphism bundle is isomorphic (as a $C^\infty(M)$-bimodule) to the module of right module maps $\mathrm{hom}_{C^\infty(M)}(\Gamma^\infty(E), \Gamma^\infty(E'))$ (cf. (4.1.32c)); the latter are linear maps $L : \Gamma^\infty(E) \to \Gamma^\infty(E')$ which satisfy additionally the right $C^\infty(M)$-linearity condition

$$L(v\, a) = L(v)\, a \ , \tag{4.2.23}$$

for all $v \in \Gamma^\infty(E)$ and $a \in C^\infty(M)$.

Our goal now is to describe the analog of homomorphism bundles in our noncommutative and nonassociative framework. Given two modules $V_\star = A_\star^n$ and $W_\star = A_\star^m$, we first consider the vector space of linear maps $\hom_F(V_\star, W_\star)$ from $V_\star$ to $W_\star$. This vector space comes together with a natural H_F-action $\triangleright : H_F \otimes \hom_F(V_\star, W_\star) \to \hom_F(V_\star, W_\star)$ given by the adjoint action

$$h \triangleright L := \big(h_{(1)_F} \triangleright \cdot\,\big) \circ L \circ \big(S_F(h_{(2)_F}) \triangleright \cdot\,\big) \ , \tag{4.2.24}$$

for all $h \in H_F$ and $L \in \hom_F(V_\star, W_\star)$. It is important to stress that we *do not* require the linear maps $L : V_\star \to W_\star$ to preserve the H_F-action. As explained in Chapter 2, this would lead to an overly rigid framework for studying noncommutative and nonassociative geometry.

The standard operations of evaluating linear maps $\hom_F(V_\star, W_\star)$ on elements in $V_\star$ and composing or tensoring linear maps with each other are in general not compatible with the H_F-action given in (4.2.24). In particular, for generic cochain twists F we have the *non-equality*

$$h \triangleright \big(L(v)\big) \neq \big(h_{(1)_F} \triangleright L\big)\big(h_{(2)_F} \triangleright v\big) \ , \tag{4.2.25}$$

for some $h \in H$, $L \in \hom_F(V_\star, W_\star)$ and $v \in V_\star$. Using internal homomorphism techniques from category theory, one can show that there exist deformations of the evaluation, composition and tensor product operations which are compatible with the H_F-actions (cf. Subsections 2.2.7 and 2.2.12. We denote these by

$$\mathrm{ev}_F : \hom_F(V_\star, W_\star) \otimes_\star V_\star \longrightarrow W_\star \ , \tag{4.2.26a}$$

$$\bullet_F : \hom_F(W_\star, X_\star) \otimes_\star \hom_F(V_\star, W_\star) \longrightarrow \hom_F(V_\star, X_\star) \ , \tag{4.2.26b}$$

$$\circledast_F : \hom_F(V_\star, X_\star) \otimes_\star \hom_F(W_\star, Y_\star) \longrightarrow \hom_F(V_\star \otimes_\star W_\star, X_\star \otimes_\star Y_\star) \ . \tag{4.2.26c}$$

The $\star$-tensor product $V_\star \otimes_\star W_\star$ is the ordinary tensor product of vector spaces equipped with the H_F-action

$$h \triangleright (v \otimes_\star w) = \big(h_{(1)_F} \triangleright v\big) \otimes_\star \big(h_{(2)_F} \triangleright w\big) \ , \tag{4.2.27}$$

for all $h \in H_F$, $v \in V_\star$ and $w \in W_\star$. For the example of the evaluation ev_F, compatibility with the H_F-actions means that

$$h \triangleright \mathrm{ev}_F(L \otimes_\star v) = \mathrm{ev}_F\big((h_{(1)_F} \triangleright L) \otimes_\star (h_{(2)_F} \triangleright v)\big) \ , \tag{4.2.28}$$

for all $h \in H$, $L \in \hom_F(V_\star, W_\star)$ and $v \in V_\star$, which resolves the problem encountered in (4.2.25).

The H_F-compatible version of the right A-linearity condition (4.2.23) is given by the weak right $A_\star$-linearity condition

$$\mathrm{ev}_F\big(L \otimes_\star (v \star a)\big) = \mathrm{ev}_F\big((\phi_F^{(-1)} \triangleright L) \otimes_\star (\phi_F^{(-2)} \triangleright v)\big) \star (\phi_F^{(-3)} \triangleright a) \ , \tag{4.2.29}$$

for all $v \in V_\star$ and $a \in A_\star$. This formula arises from the following calculation: From Lemma 2.3.11 we have the braided left $A_\star$-linearity condition $[L, a]_\star = 0$ which we argued is the correct generalisation of right A-linearity to internal homomorphisms. Evaluating this equation on some $v \in V_\star$, we obtain

$$\mathrm{ev}_F\big([L, a]_\star \otimes_\star v\,\big) = 0 \ . \tag{4.2.30}$$

Using now the evaluation identity (2.2.58) together with the left $A_\star$-linearity of ev_F (cf. Remark 2.3.13), we can simplify this equation and obtain

$$\begin{aligned} &\mathrm{ev}_F\Big((\phi_F^{(1)} \triangleright L) \otimes_\star \big((\phi_F^{(2)} \triangleright a) \star (\phi_F^{(3)} \triangleright v\,)\big)\Big) \\ &\qquad = (\phi_F^{(1)}\, R_F^{(2)} \triangleright a) \star \mathrm{ev}_F\Big((\phi_F^{(2)}\, R_F^{(1)} \triangleright L) \otimes_\star (\phi_F^{(3)} \triangleright v)\Big) \ , \end{aligned} \tag{4.2.31}$$

for all homogeneous $a \in A$ and $v \in V$. Finally using the braided symmetry of $W_\star$ and the R-matrix axioms (2.1.103b) and cancelling an occurrence of the associator from both sides we obtain equation (4.2.29). We denote by $\hom_{A_\star}(V_\star, W_\star)$ the vector space of all linear maps $L \in \hom_F(V_\star, W_\star)$ which satisfy the condition (4.2.29). It can be shown that $\hom_{A_\star}(V_\star, W_\star)$ is an H_F-module $A_\star$-bimodule, and hence a noncommutative and nonassociative vector bundle in its own right (cf. Subsection 2.3.5). We interpret $\hom_{A_\star}(V_\star, W_\star)$ as (the module of sections of) the homomorphism bundle from $V_\star$ to $W_\star$.

As $V_\star = A_\star^n$ and $W_\star = A_\star^m$ are by assumption free $A_\star$-bimodules (as are $X_\star$ and $Y_\star$), we can make use of the corresponding bases $\{e_i\}_{i=1}^n$ and $\{e_j\}_{j=1}^m$ to find simple expressions for the homomorphisms $\hom_{A_\star}(V_\star, W_\star)$, and in particular the operations (4.2.26). In the following, we shall denote (with an abuse of notation) all bases by the same symbols.

Evaluation: Because of the weak right $A_\star$-linearity condition (4.2.29), any $L \in \hom_{A_\star}(V_\star, W_\star)$ is specified by its evaluation on the basis $\{e_i\}_{i=1}^n$ of $V_\star$. Using also the basis $\{e_j\}_{j=1}^m$ of $W_\star$, we have the expansion

$$\mathrm{ev}_F(L \otimes_\star e_i) = e_j\, L^j{}_i \ , \tag{4.2.32}$$

which allows us to characterize L in terms of an $m \times n$-matrix with coefficients given by $L^j{}_i \in A_\star$. Hence we have established an isomorphism of vector spaces

$$\hom_{A_\star}(V_\star, W_\star) \longrightarrow A_\star^{m\times n} \ , \quad L \longmapsto (L^j{}_i) \ , \tag{4.2.33}$$

which assigns to any L its matrix representation. For a generic element $v = e_i\, v^i \in V_\star$ the evaluation of $L \in \hom_{A_\star}(V_\star, W_\star)$ on v can then be expressed as

$$\begin{aligned} \mathrm{ev}_F(L \otimes_\star v) &= \mathrm{ev}_F(L \otimes_\star (e_i\, v^i)) \\ &= \mathrm{ev}_F\big((\phi_F^{(-1)} \triangleright L) \otimes_\star (\phi_F^{(-2)} \triangleright e_i)\big) \star (\phi_F^{(-3)} \triangleright v^i) \\ &= \mathrm{ev}_F\big(L \otimes_\star e_i\big) \star v^i \\ &= (e_j\, L^j{}_i) \star v^i = e_j\, (L^j{}_i \star v^i) \ . \end{aligned} \tag{4.2.34}$$

In the second step we have used (4.2.29) and $e_i\,v^i = e_i \star v^i$, which follows from H_F-invariance of the basis and normalization of the twist. The third step follows by using again H_F-invariance of the basis and also normalization of the associator.

Because the evaluation operation is compatible with the H_F-actions, it follows that

$$\mathrm{ev}_F\big((h \triangleright L) \otimes_\star e_i\big) = \mathrm{ev}_F\big(h \triangleright (L \otimes_\star e_i)\big) = h \triangleright \mathrm{ev}_F\big(L \otimes_\star e_i\big) = e_j\,\big(h \triangleright L^j{}_i\big)\ , \tag{4.2.35}$$

for all $h \in H_F$ and $L \in \hom_{A_\star}(V_\star, W_\star)$, where in the first step we have used again H_F-invariance of the basis. It follows that, by equipping $A_\star^{m\times n}$ with the componentwise H_F-action, the isomorphism (4.2.33) is an isomorphism of H_F-modules. By equipping $A_\star^{m\times n}$ further with the componentwise $A_\star$-bimodule structure, the map (4.2.33) is an isomorphism of H_F-module $A_\star$-bimodules.

Composition: Given $V_\star = A_\star^n$, $W_\star = A_\star^m$ and $X_\star = A_\star^l$, one can show by similar calculations that the composition $L' \bullet_F L \in \hom_{A_\star}(V_\star, X_\star)$ of any $L \in \hom_{A_\star}(V_\star, W_\star)$ and $L' \in \hom_{A_\star}(W_\star, X_\star)$ is given by the components

$$\mathrm{ev}_F\big((L' \bullet_F L) \otimes_\star e_i\big) = e_k\,\big(L'^k{}_j \star L^j{}_i\big)\ . \tag{4.2.36}$$

Hence the isomorphism (4.2.33) sends the composition operation $\bullet_F$ to the $\star$-matrix product

$$\star : A_\star^{l\times m} \otimes_\star A_\star^{m\times n} \longrightarrow A_\star^{l\times n}\ , \quad (L'^k{}_j) \otimes_\star (L^j{}_i) \longmapsto (L'^k{}_j \star L^j{}_i)\ . \tag{4.2.37}$$

In the special case where $V_\star = W_\star = X_\star$, it follows that the endomorphism algebra $\mathrm{end}_{A_\star}(V_\star) := \hom_{A_\star}(V_\star, V_\star)$ (with product $\bullet_F$) is isomorphic to the $\star$-matrix product algebra $A_\star^{n\times n}$.

Tensor product: Given $V_\star = A_\star^n$, $W_\star = A_\star^m$, $X_\star = A_\star^l$ and $Y_\star = A_\star^p$, one can show by similar calculations that the tensor product $L' \circledast_\star L \in \hom_{A_\star}(V_\star \otimes_\star W_\star, X_\star \otimes_\star Y_\star)$

of any $L \in \hom_{A_\star}(V_\star, X_\star)$ and $L' \in \hom_{A_\star}(W_\star, Y_\star)$ is given by the components

$$\mathrm{ev}_F\big((L \circledast_F L') \otimes_\star (e_i \otimes_\star e_j)\big) = (e_k \otimes_\star e_r)\,(L^k{}_i \star L'^r{}_j) \ . \tag{4.2.38}$$

Hence the isomorphism (4.2.33) sends the tensor product operation $\circledast_\star$ to the $\star$-outer product

$$\circledast : A_\star^{l\times n} \otimes_\star A_\star^{p\times m} \longrightarrow A_\star^{(l\,p)\times(n\,m)} \ , \quad (L^k{}_i) \otimes_\star (L'^r{}_j) \longmapsto (L^k{}_i \star L'^r{}_j) \ . \tag{4.2.39}$$

4.2.4 Form-valued homomorphism bundles

As we shall see in more detail in the next sections, many homomorphisms in differential geometry are valued in the exterior algebra of differential forms $\Omega^\sharp_\star$ on $A_\star$, i.e. they are maps $L \in \hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$ for some modules $V_\star$ and $W_\star$ where $V_\star \otimes_{A_\star} W_\star$ is the quotient of $V_\star \otimes_\star W_\star$ by the relations

$$(v \star a) \otimes_\star w = (\phi^{(1)}_F \triangleright v) \otimes_\star \big((\phi^{(2)}_F \triangleright a) \star (\phi^{(3)}_F \triangleright w)\big) \ , \tag{4.2.40}$$

for all $a \in A_\star$, $v \in V_\star$ and $w \in W_\star$. In Chapter 3 we made an identification $W_\star \otimes_{A_\star} \Omega^\sharp_\star \cong W_\star$ using the right unitor in a monoidal category wherein $\Omega^\sharp_\star$ is the unit object. This was done for formal convenience. Here we do not make this identification but rather, as is more natural in a physics context, keep the tensor product with $\Omega^\sharp_\star$ explicit.

The differential forms $\Omega^\sharp_\star$ on $A_\star$ are obtained by twisting, with respect to the cochain twist $F \in H \otimes H$, the differential forms $\Omega^\sharp(M)$ on the underlying classical manifold M: As vector spaces $\Omega^\sharp_\star = \Omega^\sharp(M)$, while the product on $\Omega^\sharp_\star$ is given by the $\star$-exterior product

$$\wedge_\star := \wedge \circ F^{-1} : \Omega^p_\star \otimes_\star \Omega^q_\star \longrightarrow \Omega^{p+q}_\star \ . \tag{4.2.41}$$

The relevant H-action on $\Omega^\sharp(M)$ is given by the Lie derivative of vector fields on forms. Similarly to (4.2.7), the (graded) noncommutativity of the $\star$-exterior product is controlled by the R-matrix,

$$\omega \wedge_\star \omega' = (-1)^{|\omega|\,|\omega'|} \left(R_F^{(2)} \triangleright \omega'\right) \wedge_\star \left(R_F^{(1)} \triangleright \omega\right) , \tag{4.2.42}$$

for all homogeneous forms $\omega, \omega' \in \Omega_\star^\sharp$. Nonassociativity is controlled as in (4.2.8) by the associator

$$\left(\omega \wedge_\star \omega'\right) \wedge_\star \omega'' = (\phi_F^{(1)} \triangleright \omega) \wedge_\star \left((\phi_F^{(2)} \triangleright \omega') \wedge_\star (\phi_F^{(3)} \triangleright \omega'')\right) , \tag{4.2.43}$$

for all $\omega, \omega', \omega'' \in \Omega_\star^\sharp$. The differential

$$\mathrm{d} : \Omega_\star^p \longrightarrow \Omega_\star^{p+1} \tag{4.2.44}$$

on $\Omega_\star^\sharp$ is given by the ordinary de Rham exterior derivative and it satisfies the graded Leibniz rule

$$\mathrm{d}(\omega \wedge_\star \omega') = \mathrm{d}\omega \wedge_\star \omega' + (-1)^{|\omega|}\, \omega \wedge_\star \mathrm{d}\omega' , \tag{4.2.45}$$

for all homogeneous forms $\omega, \omega' \in \Omega_\star^\sharp$. (Note that d here satisfies the properties of $\mathrm{d}(1)$ in the categorical formalism described in Chapter 3.)

Because $\Omega_\star^\sharp$ is a graded H_F-module algebra and not only an H_F-module $A_\star$-bimodule, the modules of homomorphisms $\hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega_\star^\sharp)$ may be equipped with additional structures, which we shall now briefly describe. For this, we introduce the notation

$$V_\star^\sharp := V_\star \otimes_{A_\star} \Omega_\star^\sharp \tag{4.2.46}$$

to denote the tensor product of the module $V_\star$ with the module of differential forms $\Omega_\star^\sharp$. A generic element in $V_\star^\sharp$ is of the form $e_i \otimes_{A_\star} \omega^i$, where $\omega^i \in \Omega_\star^\sharp$. Notice that $V_\star^\sharp$ is a graded module, with $V_\star^p = V_\star \otimes_{A_\star} \Omega_\star^p$. Because $\Omega_\star^\sharp$ is a graded H_F-module algebra, $V_\star^\sharp$ is moreover a graded H_F-module $\Omega_\star^\sharp$-bimodule with left and right $\Omega_\star^\sharp$-action given by the $\star$-exterior product, i.e.

$$(e_i \otimes_{A_\star} \omega^i) \wedge_\star \omega' := e_i \otimes_{A_\star} \big(\omega^i \wedge_\star \omega'\big) \ , \tag{4.2.47a}$$

$$\omega' \wedge_\star (e_i \otimes_{A_\star} \omega^i) := e_i \otimes_{A_\star} \big(\omega' \wedge_\star \omega^i\big) \ , \tag{4.2.47b}$$

for all $\omega^i, \omega' \in \Omega^\sharp_\star$. (Notice that this definition uses H_F-invariance of the basis e_i.)

We shall now show that the module of homomorphisms $\hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$ is isomorphic (as an H_F-module $A_\star$-bimodule) to the module $\hom_{\Omega^\sharp_\star}(V^\sharp_\star, W^\sharp_\star)$ of weak right $\Omega^\sharp_\star$-linear maps, which is characterized by the condition (compare with (4.2.29))

$$\begin{aligned} &\mathrm{ev}_F\Big(L \otimes_\star \big((e_i \otimes_{A_\star} \omega^i) \ \wedge_\star \ \omega'\big)\Big) = \\ &\qquad \mathrm{ev}_F\Big((\phi_F^{(-1)} \triangleright L) \otimes_\star \big(e_i \otimes_{A_\star} (\phi_F^{(-2)} \triangleright \omega^i)\big)\Big) \ \wedge_\star (\phi_F^{(-3)} \triangleright \omega') \ , \end{aligned} \tag{4.2.48}$$

for all $\omega^i, \omega' \in \Omega^\sharp_\star$. In fact, following the same arguments as before, we use the bases of $V_\star = A^n_\star$ and $W_\star = A^m_\star$ to show that there is an isomorphism of H_F-module $\Omega^\sharp_\star$-bimodules

$$\hom_{\Omega^\sharp_\star}(V^\sharp_\star, W^\sharp_\star) \longrightarrow {\Omega^\sharp_\star}^{m\times n} \ , \quad L \longmapsto (L^j{}_i) \ . \tag{4.2.49}$$

The matrix coefficients are defined by

$$\mathrm{ev}_F\big(L \otimes_\star (e_i \otimes_{A_\star} 1)\big) = e_j \otimes_{A_\star} L^j{}_i \ , \tag{4.2.50}$$

where $1 \in A_\star \subseteq \Omega^\sharp_\star$ is the unit element. Any element $L \in \hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$ has exactly the same expansion in the bases of $V_\star$ and $W_\star$, hence we can define an isomorphism

$$(\,\cdot\,)^\sharp : \hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star) \longrightarrow \hom_{\Omega^\sharp_\star}(V^\sharp_\star, W^\sharp_\star) \tag{4.2.51}$$

by going via the matrix representations.

Given $V_\star = A^n_\star$, $W_\star = A^m_\star$ and $X_\star = A^l_\star$, we use the isomorphisms (4.2.51) and

(4.2.49) to define a composition operation

$$\bullet_F : \hom_{A_\star}(W_\star, X_\star \otimes_{A_\star} \Omega^\sharp_\star) \otimes_\star \hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star) \longrightarrow \hom_{A_\star}(V_\star, X_\star \otimes_{A_\star} \Omega^\sharp_\star) \tag{4.2.52a}$$

in terms of the $\wedge_\star$-matrix product

$$\wedge_\star : {\Omega^\sharp_\star}^{l\times m} \otimes_{A_\star} {\Omega^\sharp_\star}^{m\times n} \longrightarrow {\Omega^\sharp_\star}^{l\times n} \ , \quad ({L'}^k{}_j) \otimes_{A_\star} (L^j{}_i) \longmapsto ({L'}^k{}_j \wedge_\star L^j{}_i) \ . \tag{4.2.52b}$$

Given $V_\star = A^n_\star$, $W_\star = A^m_\star$, $X_\star = A^l_\star$ and $Y_\star = A^p_\star$, we define a tensor product operation

$$\begin{aligned} \circledast_\star : \hom_{A_\star}(V_\star, X_\star \otimes_{A_\star} \Omega^\sharp_\star) \otimes_\star \hom_{A_\star}(W_\star, Y_\star \otimes_{A_\star} \Omega^\sharp_\star) \longrightarrow \\ \hom_{A_\star}\big(V_\star \otimes_{A_\star} W_\star, (X_\star \otimes_\star Y_\star) \otimes_{A_\star} \Omega^\sharp_\star\big) \end{aligned} \tag{4.2.53a}$$

in terms of the $\wedge_\star$-outer product

$$\circledast : {\Omega^\sharp_\star}^{l\times n} \otimes_{A_\star} {\Omega^\sharp_\star}^{p\times m} \longrightarrow {\Omega^\sharp_\star}^{(l\,p)\times(n\,m)} \ , \quad (L^k{}_i) \otimes_{A_\star} ({L'}^r{}_j) \longmapsto (L^k{}_i \wedge_\star {L'}^r{}_j) \ . \tag{4.2.53b}$$

These operations generalize (4.2.37) and (4.2.39) to form-valued homomorphisms.

4.3 Nonassociative connections and curvature

4.3.1 Connections

A nonassociative connection on a module $V_\star$ is a linear map $\nabla \in \hom_F(V_\star, V_\star \otimes_{A_\star} \Omega^1_\star)$ which satisfies the Leibniz rule

$$\mathrm{ev}_F\big(\nabla \otimes_\star (v \star a)\big) = \mathrm{ev}_F\big((\phi^{(-1)}_F \triangleright \nabla) \otimes_\star (\phi^{(-2)}_F \triangleright v)\big) \star (\phi^{(-3)}_F \triangleright a) \ + \ v \otimes_\star \mathrm{d}\, a \ , \tag{4.3.1}$$

for all $v \in V_\star$ and $a \in A_\star$, where d is the exterior derivative of the differential calculus $\Omega_\star^\sharp$. Again equation (4.3.1) follows by a similar calculation to that in (4.2.31) from the condition $[L,a]_\star = \widehat{l}_\star(\mathrm{ev}_\star(\mathrm{d}(1) \otimes_\star a))$ in Lemma 3.3.2 for a connection $\nabla = (L,1) \in \mathrm{con}(V_\star)$. One also has to use item (i) of Lemma 2.1.26 together with the braided symmetry of $V_\star \otimes_{A_\star} \Omega_\star^\sharp$ and the the braided symmetry of $V_\star$ (viewed as a right $\Omega_\star^\sharp$-module) and $\mathrm{ev}\big(\rho_{\Omega_\star^1}(\beta) \circ \mathrm{d}(1) \otimes a\big) = \mathrm{d}\, a$.

We denote the space of connections on $V_\star$ by $\mathrm{con}_F(V_\star)$ and note that it is an affine space over the module of homomorphisms $\mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega_\star^1)$.

As $V_\star = A_\star^n$ is by assumption a free $A_\star$-bimodule, we can describe any connection $\nabla \in \mathrm{con}_F(V_\star)$ in terms of its coefficients $\Gamma^j{}_i \in \Omega_\star^1$ defined by

$$\mathrm{ev}_F(\nabla \otimes_\star e_i) =: e_j \otimes_{A_\star} \Gamma^j{}_i \ . \tag{4.3.2}$$

Using (4.3.1), after a short calculation we obtain

$$\mathrm{ev}_F(\nabla \otimes_\star v) = e_i \otimes_{A_\star} \big(\mathrm{d}v^i \,+\, \Gamma^i{}_j \star v^j\big) \ , \tag{4.3.3}$$

for all $v = e_i\, v^i \in V_\star$.

As $\mathrm{con}_F(V_\star) \subseteq \mathrm{hom}_F(V_\star, V_\star \otimes_{A_\star} \Omega_\star^1)$ is an affine subspace, we can act with any $h \in H_F$ on a connection ∇ and obtain an element $h \triangleright \nabla \in \mathrm{hom}_F(V_\star, V_\star \otimes_{A_\star} \Omega_\star^1)$, which however in general does not lie in $\mathrm{con}_F(V_\star)$: In contrast to the Leibniz rule (4.3.3), $h \triangleright \nabla$ satisfies

$$\mathrm{ev}_F\big((h \triangleright \nabla) \otimes_\star v\big) = e_i \otimes_{A_\star} \big(\epsilon_F(h)\, \mathrm{d}v^i \,+\, (h \triangleright \Gamma^i{}_j) \star v^j\big) \ , \tag{4.3.4}$$

for all $v = e_i\, v^i \in V_\star$. In particular, if $h \in H_F$ satisfies $\epsilon_F(h) = 1$ then $h \triangleright \nabla \in \mathrm{con}_F(V_\star)$, while if $\epsilon_F(h) = 0$ then $h \triangleright \nabla \in \mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega_\star^1)$.

Similarly to the case of homomorphisms (4.2.51), we can lift connections $\nabla \in \mathrm{con}_F(V_\star)$ to linear maps $\nabla^\sharp \in \mathrm{end}_F(V_\star^\sharp)$, which then satisfy the condition

$$\mathrm{ev}_F\big(\nabla^\sharp \otimes_\star (e_i \otimes_{A_\star} \omega^i)\big) = e_i \otimes_{A_\star} \big(\mathrm{d}\omega^i \,+\, \Gamma^i{}_j \wedge_\star \omega^j\big) \ , \tag{4.3.5}$$

for all $\omega^i \in \Omega^\sharp_\star$. Notice that (4.3.5) implies the graded Leibniz rule

$$\mathrm{ev}_F\big(\nabla^\sharp \otimes_\star (s \wedge_\star \omega')\big) = \\ \mathrm{ev}_F\big((\phi_F^{(-1)} \triangleright \nabla^\sharp) \otimes_\star (\phi_F^{(-2)} \triangleright s)\big) \wedge_\star (\phi_F^{(-3)} \triangleright \omega') + (-1)^{|s|}\, s \wedge_\star \mathrm{d}\omega' \ , \quad (4.3.6)$$

for all homogeneous forms $s = e_i \otimes_\star \omega^i \in V^\sharp_\star$ and $\omega' \in \Omega^\sharp_\star$. This follows from the calculation

$$\begin{aligned} &\mathrm{ev}_F\big(\nabla^\sharp \otimes_\star (s \wedge_\star \omega')\big) \\ &= e_i \otimes_{A_\star} \big(\mathrm{d}(\omega^i \wedge_\star \omega') + \Gamma^i{}_j \wedge_\star (\omega^j \wedge_\star \omega')\big) \\ &= e_i \otimes_{A_\star} \big(\mathrm{d}\omega^i \wedge_\star \omega' + \omega^i \wedge_\star \mathrm{d}\omega' + \big((\phi_F^{(-1)} \triangleright \Gamma^i{}_j) \wedge_\star (\phi_F^{(-2)} \triangleright \omega^j)\big) \wedge_\star (\phi_F^{(-3)} \triangleright \omega')\big) \\ &= \mathrm{ev}_F\big((\phi_F^{(-1)} \triangleright \nabla^\sharp) \otimes_\star (\phi_F^{(-2)} \triangleright s)\big) \wedge_\star (\phi_F^{(-3)} \triangleright \omega') + (-1)^{|s|}\, s \wedge_\star \mathrm{d}\omega' \ . \end{aligned} \quad (4.3.7)$$

The first equality follows from (4.3.3), the second equality follows from (4.2.45) and (4.2.43), and the third equality follows from (4.3.4) together with the normalisation of the associator.

4.3.2 Connections on tensor products

Given $V_\star = A^n_\star$ and $W_\star = A^m_\star$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$ and $\nabla' \in \mathrm{con}_F(W_\star)$, we can construct a connection on $V_\star \otimes_{A_\star} W_\star$ by taking their sum $\nabla \boxplus_F \nabla'$ (cf. Subsection 3.3.2). In terms of the coefficients $\Gamma^k{}_i, \Gamma'^l{}_j \in \Omega^1_\star$, the sum of connections takes a simple form and it is specified by the coefficients

$$\mathrm{ev}_F\big((\nabla \boxplus_F \nabla') \otimes_\star (e_i \otimes_{A_\star} e_j)\big) = (e_k \otimes_{A_\star} e_l) \otimes_{A_\star} \big(\Gamma^k{}_i\, \delta^l{}_j + \delta^k{}_i\, \Gamma'^l{}_j\big) \ . \quad (4.3.8)$$

On a generic element $v \otimes_{A_\star} w = e_i \otimes_{A_\star} e_j\, (v^i \star w^j) \in V_\star \otimes_{A_\star} W_\star$, the sum of connections acts as

$$\mathrm{ev}_F\big((\nabla \boxplus_F \nabla') \otimes_\star (v \otimes_{A_\star} w)\big) = \\ (e_k \otimes_{A_\star} e_l) \otimes_{A_\star} \big(\mathrm{d}(v^k \star w^l) + \Gamma^k{}_i \wedge_\star (v^i \star w^l) + \Gamma'^l{}_j \wedge_\star (v^k \star w^j)\big) \ . \quad (4.3.9)$$

This follows directly from (4.3.3) with (4.3.8) for the connection $\nabla \boxplus_F \nabla'$.

The sum of connections can be consistently extended to tensor products of finitely many modules by inductively using (4.3.8). For example, given $V_\star = A_\star^n$, $W_\star = A_\star^m$ and $X_\star = A_\star^l$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$, $\nabla' \in \mathrm{con}_F(W_\star)$ and $\nabla'' \in \mathrm{con}_F(X_\star)$, then $(\nabla \boxplus_F \nabla') \boxplus_F \nabla'' \in \mathrm{con}_F((V \otimes_{A_\star} W_\star) \otimes_{A_\star} X_\star)$ is specified by the connection coefficients

$$\begin{aligned}&\mathrm{ev}_F\Big(\big((\nabla \boxplus_F \nabla') \boxplus_F \nabla''\big) \otimes_\star \big((e_i \otimes_{A_\star} e_j) \otimes_{A_\star} e_k\big)\Big) = \\ &\big((e_{i'} \otimes_{A_\star} e_{j'}) \otimes_{A_\star} e_{k'}\big) \otimes_{A_\star} \big(\Gamma^{i'}{}_i \, \delta^{j'}{}_j \, \delta^{k'}{}_k + \delta^{i'}{}_i \, \Gamma'^{\,j'}{}_j \, \delta^{k'}{}_k + \delta^{i'}{}_i \, \delta^{j'}{}_j \, \Gamma''^{\,k'}{}_k\big) \ . \qquad (4.3.10)\end{aligned}$$

Moreover, $(\nabla \boxplus_F \nabla') \boxplus_F \nabla''$ and $\nabla \boxplus_F (\nabla' \boxplus_F \nabla'')$ are related by adjoining the associator

$$(\nabla \boxplus_F \nabla') \boxplus_F \nabla'' = \phi_F^{-1} \circ \big(\nabla \boxplus_F (\nabla' \boxplus_F \nabla'')\big) \circ \phi_F \ . \qquad (4.3.11)$$

4.3.3 Connections on homomorphism bundles

Given $V_\star = A_\star^n$ and $W_\star = A_\star^m$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$ and $\nabla' \in \mathrm{con}_F(W_\star)$, we can construct a connection on $\hom_{A_\star}(V_\star, W_\star)$ by taking their adjoint $\mathrm{ad}_{\bullet F}(\nabla', \nabla)$ (cf. 3.3.3). In terms of the coefficients $\Gamma^k{}_i, \Gamma'^{\,l}{}_j \in \Omega^1_\star$, the adjoint connection takes a simple form: Denoting by $\{e_j{}^i\}$ the basis of $\hom_{A_\star}(V_\star, W_\star)$ given by the isomorphism (4.2.33) and the standard basis of $A_\star^{m\times n}$, the coefficients of $\mathrm{ad}_{\bullet F}(\nabla', \nabla)$ are given by

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet F}(\nabla', \nabla) \otimes_\star e_j{}^i\big) = e_{j'}{}^{i'} \otimes_{A_\star} \big(\Gamma'^{\,j'}{}_j \, \delta^i{}_{i'} - \delta^{j'}{}_j \, \Gamma^i{}_{i'}\big) \ . \qquad (4.3.12)$$

This follows from the calculation

$$\begin{aligned}\mathrm{ev}_F\big(\mathrm{ad}_{\bullet F}(\nabla', \nabla) \otimes_\star e_j{}^i\big) &= \nabla' \bullet e_j{}^i - e_j{}^i \bullet \nabla \\ &= \mathrm{ev}(\nabla' \otimes \mathrm{ev}(e_j{}^i \otimes -)) - \mathrm{ev}(e_j{}^i \otimes \mathrm{ev}(\nabla \otimes -)) \\ &= e_{j'}{}^{i'} \otimes_{A_\star} \big(\Gamma'^{\,j'}{}_j \, \delta^i{}_{i'} - \delta^{j'}{}_j \, \Gamma^i{}_{i'}\big) \ , \qquad (4.3.13)\end{aligned}$$

using the definition of $\mathrm{ad}_{\bullet_F}$ and $\bullet_F$ together with the H-invariance of the standard basis and the normalisation of the R-matrix and associator, and (4.3.2) with the definition of the basis $\{e_j{}^i\}$ in the final step.

On a generic element $L = e_j{}^i\, L^j{}_i \in \mathrm{hom}_{A_\star}(V_\star, W_\star)$, the adjoint connection acts as

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla', \nabla) \otimes_\star L\big) = \\ e_{j'}{}^{i'} \otimes_{A_\star} \big(\mathrm{d}L^{j'}{}_{i'} + \Gamma'^{\,j'}{}_j \star L^j{}_{i'} - (R_F^{(2)} \triangleright L^{j'}{}_i) \star (R_F^{(1)} \triangleright \Gamma^i{}_{i'})\big) \ , \tag{4.3.14}$$

where in the last term we have used the R-matrix to rearrange the term $\Gamma^i{}_{i'} \star L^{j'}{}_i$ so that $\star$-matrix multiplication is obvious. This follows directly from (4.3.3) with (4.3.12) for the connection $\mathrm{ad}_{\bullet_F}(\nabla', \nabla)$.

For any two vector bundles $V_\star$ and $W_\star$ the adjoint connection $\mathrm{ad}_{\bullet_F}$ extends to form-valued homomorphisms $L \in \mathrm{hom}_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$. The resulting expression

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla', \nabla) \otimes_\star L\big) = \\ e_{j'}{}^{i'} \otimes_{A_\star} \big(\mathrm{d}L^{j'}{}_{i'} + \Gamma'^{\,j'}{}_j \wedge_\star L^j{}_{i'} - (-1)^{|L|}\, (R_F^{(2)} \triangleright L^{j'}{}_i) \wedge_\star (R_F^{(1)} \triangleright \Gamma^i{}_{i'})\big) \tag{4.3.15}$$

is very similar to (4.3.14) whereby we simply replace $\star$-products by $\wedge_\star$-products and include a degree-dependent sign factor in front of the last term.

4.3.4 Curvature

The curvature of a connection $\nabla \in \mathrm{con}_F(V_\star)$ is given by the graded R-matrix commutator

$$R(\nabla) := \tfrac{1}{2}\, [\nabla^\sharp, \nabla^\sharp]_F := \tfrac{1}{2}\, \big(\nabla^\sharp \bullet_F \nabla^\sharp + (R_F^{(2)} \triangleright \nabla^\sharp) \bullet_F (R_F^{(1)} \triangleright \nabla^\sharp)\big) \tag{4.3.16}$$

of its lift $\nabla^\sharp \in \mathrm{end}_F(V^\sharp_\star)$ defined in (4.3.5). Due to the graded Leibniz rule (4.3.6), it follows that $R(\nabla) \in \mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^2_\star)$ is a homomorphism valued in 2-forms.

The coefficients of the curvature are given by

$$\mathrm{ev}_F\big(R(\nabla)\otimes_\star e_i\big) = e_j\otimes_{A_\star} R^j{}_i = e_j\otimes_{A_\star}\left(\mathrm{d}\Gamma^j{}_i\,+\,\tfrac{1}{2}\,[\Gamma\,,\,\Gamma]_\star{}^j{}_i\,\right)\,, \tag{4.3.17a}$$

where

$$[\Gamma\,,\,\Gamma]_\star{}^j{}_i := \Gamma^j{}_k\wedge_\star\Gamma^k{}_i + \big(R_F^{(2)}\triangleright\Gamma^j{}_k\big)\wedge_\star\big(R_F^{(1)}\triangleright\Gamma^k{}_i\big)\ . \tag{4.3.17b}$$

This follows from (4.3.2), (4.3.3) and (4.3.5) together with Proposition 2.2.13 (ii).

On the sum of connections $\nabla\in\mathrm{con}_F(V_\star)$ and $\nabla'\in\mathrm{con}_F(W_\star)$, the curvature $R(\nabla\boxplus_F\nabla')$ has the desired additive behavior

$$\mathrm{ev}_F\big(R(\nabla\boxplus_F\nabla')\otimes_\star(e_i\otimes_{A_\star}e_j)\big) = (e_k\otimes_{A_\star}e_l)\otimes_{A_\star}\big(R^k{}_i\,\delta^l{}_j\,+\,\delta^k{}_i\,R'^l{}_j\big)\ . \tag{4.3.18}$$

This follows from the result $R(\nabla\boxplus_F\nabla') = R(\nabla)\circledast_F 1 + 1\circledast_F R(\nabla'))$ (cf. Proposition 3.4.5) together with (4.2.38).

The Bianchi tensor of a connection $\nabla\in\mathrm{con}_F(V_\star)$ is defined by acting with the adjoint connection on the curvature using (4.3.15) to get

$$\mathrm{Bianchi}(\nabla) := \mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla,\nabla)\otimes_\star R(\nabla)\big)\ . \tag{4.3.19}$$

By definition, it follows that $\mathrm{Bianchi}(\nabla)\in\mathrm{hom}_{A_\star}(V_\star, V_\star\otimes_{A_\star}\Omega^3_\star)$ is a homomorphism valued in 3-forms. Using (4.3.15) we find

$$\mathrm{ev}_F\big(\mathrm{Bianchi}(\nabla)\otimes_\star e_i\big) = e_j\otimes_{A_\star}\mathrm{Bianchi}^j{}_i = e_j\otimes_{A_\star}\big(\mathrm{d}R^j{}_i + [\Gamma, R]_\star{}^j{}_i\big)\ , \tag{4.3.20a}$$

where

$$[\Gamma, R]_\star{}^j{}_i := \Gamma^j{}_k\,\wedge_\star\,R^k{}_i\,-\,\big(R_F^{(2)}\triangleright R^j{}_k\big)\,\wedge_\star\,\big(R_F^{(1)}\triangleright\Gamma^k{}_i\big)\ . \tag{4.3.20b}$$

This follows by a simple calculation using (4.3.15).

An interesting consequence of the noncommutativity and nonassociativity of $A_\star$

(which is controlled by the R-matrix and associator) is that in general the Bianchi tensor does not vanish, i.e. the Bianchi identity is generally violated. However, for trivial R-matrix and associator we recover the usual Bianchi identity in classical differential geometry for any connection ∇.

4.4 Nonassociative field theory

4.4.1 Yang-Mills theory

Let M be an oriented m-dimensional manifold equipped with an H-invariant Riemannian or Lorentzian metric. Then the classical Hodge operator $*_M : \Omega^p(M) \to \Omega^{m-p}(M)$ is H-equivariant, i.e. $*_M \circ (h \triangleright \cdot\,) = (h \triangleright \cdot\,) \circ *_M$ for all $h \in H$. We equip the deformed differential forms with the same Hodge operator, leading to an H_F-equivariant map

$$*_M : \Omega^p_\star \longrightarrow \Omega^{m-p}_\star \ . \tag{4.4.1}$$

Given any module $V_\star = A^n_\star$ and any connection $\nabla \in \mathrm{con}_F(V_\star)$, let $\mathcal{L}(\nabla) \in \hom_{A_\star}(V_\star, V_\star \otimes_\star \Omega^m_\star)$ be the homomorphism valued in top-forms which is given by the components

$$\mathcal{L}^j{}_i = \tfrac{1}{2}\, F^j{}_k \wedge_\star *_M F^k{}_i \ , \tag{4.4.2}$$

where as usual we denote the curvature of a gauge connection by $F^j{}_i = \mathrm{d}\Gamma^j{}_i + \frac{1}{2}\,[\Gamma\,,\,\Gamma]_\star{}^j{}_i$. The action functional for Yang-Mills gauge theory is given by tracing and integrating $\mathcal{L}(\nabla)$, i.e.

$$S_{\mathrm{YM}}(\nabla) := \int_M \mathrm{Tr}\big(\mathcal{L}(\nabla)\big) = \frac{1}{2} \int_M F^j{}_k \wedge_\star *_M F^k{}_j \ . \tag{4.4.3}$$

We shall now show that, under certain natural conditions on the twist $F \in H \otimes H$ and the connection ∇, the Yang-Mills action (4.4.3) is real-valued.

The first condition is that F is Hermitean, i.e. it defines a Hermitean star-product on $A_\star$. This means that $(a \star b)^* = b^* \star a^*$, where * denotes the involution given by pointwise complex conjugation of functions on M. This is clearly the case for Examples 4.2.1 and 4.2.2 (indeed $F^* = F_{21}$ for the twist in both examples due to the antisymmetry of Θ^{ij} resp. R^{ijk}). We extend the involution * on $A_\star$ to a graded involution on the differential forms $\Omega^\sharp_\star$ by setting

$$(\omega \wedge_\star \omega')^* = (-1)^{|\omega|\,|\omega'|}\ \omega'^{\,*} \wedge_\star \omega^* \ , \quad (\mathrm{d}\omega)^* = \mathrm{d}\omega^* \ , \tag{4.4.4}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$.

The second condition is that ∇ is unitary, i.e. the corresponding connection coefficients satisfy

$$\Gamma^j{}_i{}^* = -\Gamma^i{}_j \ . \tag{4.4.5}$$

Using (4.3.17) one easily shows that the curvature of a unitary connection is an anti-Hermitean matrix, i.e.

$$F^j{}_i{}^* = -F^i{}_j \ . \tag{4.4.6}$$

The third condition is the graded 2-cyclicity property

$$\int_M \omega \ \wedge_\star \ \omega' = (-1)^{|\omega|\,|\omega'|} \int_M \omega' \ \wedge_\star \ \omega \ , \tag{4.4.7}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$. This property holds for Abelian twists, as in Example 4.2.1, and also for the nonassociative deformation of Example 4.2.2, see [27].

The first two conditions imply that the complex conjugate of the action (4.4.3) can be simplified as

$$
\begin{aligned}
S_{\mathrm{YM}}(\nabla)^* &= \frac{1}{2}\int_M \big(F^j{}_k \wedge_\star *_M F^k{}_j\big)^* \\
&= \frac{1}{2}\int_M *_M F^k{}_j{}^* \wedge_\star F^j{}_k{}^* \\
&= \frac{1}{2}\int_M *_M F^j{}_k \wedge_\star F^k{}_j \ , \qquad (4.4.8)
\end{aligned}
$$

where in the second step we have also used compatibility between the Hodge operator and the complex conjugation involution. The third condition then implies that we can interchange the two terms in the last equality of (4.4.8), and hence find that the noncommutative and nonassociative Yang-Mills action is real, i.e.

$$
S_{\mathrm{YM}}(\nabla)^* = S_{\mathrm{YM}}(\nabla) \ . \qquad (4.4.9)
$$

In particular, the noncommutative and nonassociative Yang-Mills action (4.4.3) is real-valued for all unitary connections in Examples 4.2.1 and 4.2.2.

4.4.2 Einstein-Cartan gravity

The field content of Einstein-Cartan gravity is a spin connection ∇ and a vielbein field E. Let M be an oriented m-dimensional manifold which admits a trivial Dirac spinor bundle

$$
S = M \times \mathbb{C}^{2^{\lfloor \frac{m}{2} \rfloor}} \longrightarrow M \ . \qquad (4.4.10)
$$

We denote the module of sections of the spinor bundle by $V := \Gamma^\infty(S) = A^{2^{\lfloor \frac{m}{2} \rfloor}}$.

Without loss of generality, here we can take $H = U\mathrm{Vec}(M)$ to be the Hopf algebra of all infinitesimal diffeomorphisms of M. Then given any cochain twist $F \in H \otimes H$, we twist $A = C^\infty(M)$ to a noncommutative and nonassociative algebra $A_\star$ and V to an H_F-module $A_\star$-bimodule $V_\star = A_\star^{2^{\lfloor \frac{m}{2} \rfloor}}$.

A spin connection on $V_\star$ is a connection $\nabla \in \mathrm{con}_F(V_\star)$ for which the coefficients take the special form

$$\Gamma^j{}_i = \tfrac{1}{4}\,\omega^{ab}\,\gamma_{ab}{}^j{}_i \ , \tag{4.4.11}$$

where $\omega^{ab} \in \Omega^1_\star$ is antisymmetric in ab and $\gamma_{ab} = \frac{1}{2}\,[\gamma_a, \gamma_b]$ is given by the commutator of the gamma-matrices γ_a; here the indices $a, b, \dots$ run from 1 to m, the dimension of M, while $i, j, \dots$ run from 1 to $2^{\lfloor \frac{m}{2} \rfloor}$, the rank of the Dirac spinor bundle S. The curvature (4.3.17) of a spin connection can be computed with some standard gamma-matrix algebra and it reads as

$$R^j{}_i = \tfrac{1}{4}\,R^{ab}\,\gamma_{ab}{}^j{}_i = \tfrac{1}{4}\,\big(\mathrm{d}\omega^{ab} \,+\, \omega^a{}_c \wedge_\star \omega^{cb}\big)\,\gamma_{ab}{}^j{}_i \ , \tag{4.4.12}$$

where the c-index was lowered by the flat metric η_{ab}. (This follows from properties of the gamma matrices and the antisymmetry of ω^{ab}.)

A vielbein is a homomorphism $E \in \hom_{A_\star}(V_\star, V_\star \otimes_\star \Omega^1_\star)$ valued in 1-forms for which the coefficients take the special form

$$E^j{}_i = E^a\,\gamma_a{}^j{}_i \ , \tag{4.4.13}$$

where $E^a \in \Omega^1_\star$.

Let us assume for the moment that the dimension m of M is even. We propose the noncommutative and nonassociative generalization of the Einstein-Cartan action functional given by

$$S^{\mathrm{even}}_{\mathrm{EC}}(\nabla, E) := \int_M \Big(E_{\mathrm{left}}^{a_1\cdots a_{\frac{m}{2}-1}} \wedge_\star R^{a_{\frac{m}{2}}\,a_{\frac{m}{2}+1}}\Big) \wedge_\star E_{\mathrm{right}}^{a_{\frac{m}{2}+2}\cdots a_m}\ \epsilon_{a_1\cdots a_m} \ , \tag{4.4.14}$$

where $\epsilon_{a_1\cdots a_m}$ is the antisymmetric tensor and

$$E_{\mathrm{left}}^{a_1\cdots a_k} := \big(\cdots\big((E^{[a_1} \wedge_\star E^{a_2}) \wedge_\star E^{a_3}\big)\cdots\big) \wedge_\star E^{a_k]} \ , \tag{4.4.15a}$$

$$E_{\mathrm{right}}^{a_1\cdots a_k} := E^{[a_1} \wedge_\star \big(\cdots\big(E^{a_{k-2}} \wedge_\star (E^{a_{k-1}} \wedge_\star E^{a_k]})\big)\cdots\big) \ , \tag{4.4.15b}$$

is the $\wedge_\star$-product of k vielbeins in $\Omega^k_\star$ with special bracketing conventions and totally antisymmetrized (with weight 1) in the indices $a_1 \cdots a_k$. This choice of bracketing allows us to show that the Einstein-Cartan action (4.4.14) is real-valued, under similar assumptions as for the Yang-Mills action.

Let us now assume that the twist F is Hermitean and further demand the reality conditions

$$\omega^{ab*} = -\omega^{ba} = \omega^{ab} \ , \quad E^{a*} = E^a \ , \tag{4.4.16}$$

for the spin connection and vielbein. As a consequence, we obtain

$$R^{ab*} = -R^{ba} = R^{ab} \ , \quad E_{\rm left}^{a_1\cdots a_k\,*} = E_{\rm right}^{a_1\cdots a_k} \ . \tag{4.4.17}$$

The complex conjugate of the action (4.4.14) can now be simplified as

$$\begin{aligned} S_{\rm EC}^{\rm even}(\nabla,E)^* &= (-1)^{\frac{m}{2}-1}\int_M E_{\rm left}^{a_{\frac{m}{2}+2}\cdots a_m}\wedge_\star\Big(R^{a_{\frac{m}{2}}a_{\frac{m}{2}+1}}\wedge_\star E_{\rm right}^{a_1\cdots a_{\frac{m}{2}-1}}\Big)\,\epsilon_{a_1\cdots a_m} \\ &= \int_M E_{\rm left}^{a_1\cdots a_{\frac{m}{2}-1}}\wedge_\star\Big(R^{a_{\frac{m}{2}}a_{\frac{m}{2}+1}}\wedge_\star E_{\rm right}^{a_{\frac{m}{2}+2}\cdots a_m}\Big)\,\epsilon_{a_1\cdots a_m} \ , \end{aligned} \tag{4.4.18}$$

where the sign factor in the first equality is due to (4.4.4). In the second equality we have reordered the indices of $\epsilon_{a_1\cdots a_m}$ by using its total antisymmetry property.

We further assume the 3-cyclicity property

$$\int_M (\omega\wedge_\star\omega')\wedge_\star\omega'' = \int_M \omega\wedge_\star(\omega'\wedge_\star\omega'') \ , \tag{4.4.19}$$

for all $\omega,\omega',\omega''\in\Omega_\star^\sharp$. This property obviously holds for Abelian twists as in Example 4.2.1, because they give strictly associative deformations. For the nonassociative deformation of Example 4.2.2 the 3-cyclicity property is shown in [27]. We can then rebracket the expression after the last equality of (4.4.18) and find that the noncommutative and nonassociative Einstein-Cartan action in even dimensions (4.4.14) is real, i.e.

$$S_{\rm EC}^{\rm even}(\nabla,E)^* = S_{\rm EC}^{\rm even}(\nabla,E) \ . \tag{4.4.20}$$

In the case of an odd-dimensional manifold M, one way to obtain a real-valued Einstein-Cartan action functional is to modify (4.4.14) as

$$\begin{aligned} S^{\mathrm{odd}}_{\mathrm{EC}}(\nabla, E) := \frac{1}{2} \int_M \Big(E_{\mathrm{left}}^{a_1 \cdots a_{\frac{m-1}{2}-1}} \wedge_\star R^{a_{\frac{m-1}{2}} a_{\frac{m-1}{2}+1}} \Big) \wedge_\star E_{\mathrm{right}}^{a_{\frac{m-1}{2}+2} \cdots a_m} \, \epsilon_{a_1 \cdots a_m} \\ + \frac{1}{2} \int_M \Big(E_{\mathrm{left}}^{a_1 \cdots a_{\frac{m-1}{2}}} \wedge_\star R^{a_{\frac{m-1}{2}+1} a_{\frac{m-1}{2}+2}} \Big) \wedge_\star E_{\mathrm{right}}^{a_{\frac{m-1}{2}+3} \cdots a_m} \, \epsilon_{a_1 \cdots a_m} \ , \end{aligned} \tag{4.4.21}$$

where in the first line the form degree of E_{right} is larger by 1 than the form degree of E_{left} and vice versa in the second line. Under the same assumptions as in the even-dimensional case, one can show that the action (4.4.21) is real-valued, i.e.

$$S^{\mathrm{odd}}_{\mathrm{EC}}(\nabla, E)^* = S^{\mathrm{odd}}_{\mathrm{EC}}(\nabla, E) \ . \tag{4.4.22}$$

In fact, the second term in (4.4.21) is the conjugate of the first term and vice versa.

In particular, the noncommutative and nonassociative Einstein-Cartan gravity action in even dimensions (4.4.14) and in odd dimensions (4.4.21) is real-valued in Examples 4.2.1 and 4.2.2.

4.5 Summary

In this chapter we have applied the constructions in Chapter 2 to the concrete examples of deformation quantization of G-equivariant vector bundles over G-manifolds. In particular we constructed a functor between the category of G-manifolds and the category of commutative algebra objects in the representation category of the quasi-Hopf algebra obtained by cochain twisting the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. This clarified what we mean by a noncommutative and nonassociative space as a commutative algebra object in $[U\mathfrak{g}_F, \mathscr{M}]$. We also constructed a closed braided monoidal functor between the category of G-equivariant vector bundles over a manifold M and the category of symmetric bimodules over $C^\infty(M)_F$ the twisted function algebra on M. This clarified what we mean by a noncommutative and nonassociative vector bundle as a symmetric bimodule object in $[U\mathfrak{g}_F, \mathscr{M}]$. We also provided examples of noncommutative and nonassociative spaces which fit into this framework which include the Q and R-flux compactifications of closed string theory. Finally we considered how the constructions in Chapter 3 may be applied in the simplest model of cochain twist deformations of trivial vector bundles over noncommutative and nonassociative spaces and provided physically viable action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces. This concludes the technical part of this thesis.

Chapter 5

Conclusion

5.1 Summary and main contributions

This thesis aimed to provide a rigorous mathematical framework for noncommutative geometry that could be generalised also to nonassociative structures. Within this framework it aimed to provide an abstractly motivated procedure to lift noncommutative connections to tensor products of vector bundles and also to tensor fields. In this sense it aimed to clarify and generalise the formalism developed in [6] in an approach similar to that taken in [13] but making use of internal homomorphisms rather than morphisms for the construction of geometric entities in order to solve the problem of quantum rigidity for the configuration space of noncommutative connections. It also aimed to provide a first step towards understanding the effect of noncommutative and nonassociative deformations of spacetime geometry on models of quantum gravity.

The key insight in Chapter 2 was that, in addition to being braided monoidal, the representation category of a triangular quasi-Hopf algebra is closed. This is an important observation since it enables us to enlarge the morphisms of the representation category of a quasi-Hopf algebra by internal homomorphisms and ultimately provide sufficiently large configuration spaces for noncommutative connections with also more potential for dynamical field content. A minor technical point in this chapter was to find the correct form of the coherence map for the cochain twisting of the internal hom-structure and also, importantly, the form of the tensor product morphism for internal homomorphisms with which connections are lifted to tensor products in Chapter 3.

We saw in Chapter 2 that the internal homomorphisms carry an adjoint action of H and come equipped with a currying map from which the non-trivial structures of evaluation, composition and tensor product morphisms for internal homomorphisms are built. Regarding all geometric quantities as internal hom-objects means they can transform non-trivially under the adjoint H-action (not like morphisms which must preserve H-action) and must be evaluated, composed and tensor producted with the internal operations of evaluation, composition and tensor product which differ from the standard operations.

In Chapter 2 we aimed to make all proofs element-independent in order to see that our results are model independent and generalisable to other closed braided monoidal categories which are complete (have all limits) and cocomplete (have all colimits). This element-independent approach also made proofs simpler usually reducing them to simply manipulating the axioms of a triangular quasi-Hopf algebra and its representations. This model-independence may open possible directions for future work. We also understood twist deformation quantisation as a diagrammatic program. In particular we understood that the twist deformation quantisation functor is determined by the structures of the representation category. From this perspective we could understand the origin of the explicit expression for the currying map in the category $[H, \mathscr{M}]$ as the structure which arises under applying the coherence maps to cochain twisting the evaluation of internal homomorphisms.

To summarise, the important contributions of Chapter 2 were to show that the morphisms in $[H, \mathscr{M}]$ are contained in the internal homomorphisms, to find a tensor product operation for internal homomorphisms which was used in Chapter 3 for defining the lift of connections to tensor products, and to build a commutator for the internal endomorphism algebra of an object which endows it with the structure of a Lie algebra. This structure is used in many proofs of properties of geometric quantities in Chapter 3.

In Chapter 3 we formulated notions of classical differential geometry on one algebra object and its bimodule objects using universal constructions internal to the representation category of an arbitrary triangular quasi-Hopf algebra. Most importantly we were able to make use of the categorical formalism developed in Chapter 2 to make structurally correct definitions for the notions of connections together with their tensor product structure. Rather than simply replacing operations by $\star$-products the framework explicitly indicates where to insert instances of the associator and R-matrices. In Chapter 4 we saw that the correct formulae are indeed in general obtained simply by replacing operations by $\star$-products, but this is in the very simplest setting of trivial vector bundles and it is not expected to remain true for arbitrary vector bundles. Our formalism has nonetheless justified the replacing of operations by $\star$-products in the setting of trivial vector bundles used most often in noncommutative geometry.

To summarise, the main contributions of Chapter 3 were to provide morphisms for lifting connections to tensor products and internal homomorphisms, and also to provide a categorical description of a left-right symmetric definition of curvature.

In Chapter 4 we applied the framework developed in Chapter 3 to obtain explicit expressions for connections and their curvatures on noncommutative and nonassociative vector bundles in the simplest example of cochain twist deformations of trivial vector bundles over noncommutative and nonassociative spaces and we provided physically viable action functionals for Yang-Mills theory and Einstein-Cartan gravity. The latter was inspired by the work in [3]. In that paper extra terms had to be added to the curvature and spin connection. Using our categorical formalism we have shown that these additional terms are unnecessary. Although Chapter 4 attempted to make the work of the previous chapters in this thesis more accessible to a physics audience it was a very preliminary step in that direction.

5.2 Future work

Other possible avenues which could be explored together with projects which have already been derived from this work include:

From Chapter 2. One can view the representation category of a quasi-Hopf algebra as a duoidal category and consider further properties such as the compatibility of the internal tensor product with the action of an algebra object in this setting. The formalism also begs a generalisation to higher categorical structures.

From Chapter 3. One can view geometrical quantities such as derivations and connections as functors and consider what might be understood from this generalisation (cf. [44]). One can also consider the categorical construction of other important notions of geometry such as gauge groups and principal bundles (see below).

From Chapter 4. One may consider other types of vector bundles over noncommutative and nonassociative spaces, in particular arbitrary finitely generated and projective modules permitted by the Serre-Swan theorem for noncommutative geometry. In the case of trivial vector bundles much of the nonassociativity and noncommutativity is lost due to the H-invariance of the standard basis. One may also calculate the field equations as in [3] arising from the Yang-Mills and Einstein-Cartan actions for non-geometry after establishing the correct form of the gauge group.

Structure group and principle bundles. With the tools developed in this thesis we have no control over the structure group of the frame bundle for Einstein-Cartan gravity. In order to reduce the structure group to the correct subgroup it is necessary to have a framework capable of dealing with principle bundles. Subsequent work that has been done involves generalisations of our approach to noncommutative and nonassociative vector bundles to the case of principal bundles (i.e. Hopf-Galois extensions) (cf. [56]). An interesting problem is the correct definition of the gauge group of a noncommutative and nonassociative principal bundle. Motivated by our internal point of view, the gauge group should arise as a certain subspace of a 'mapping space', which one can formalise by using topos theoretic techniques such as those appearing in synthetic differential geometry.

Quantum rigidity or noncommutative symmetry breaking. In this thesis we have solved the problem of quantum rigidity of geometrical notions such as connections, where by quantum rigidity we mean the effect that configuration spaces of quantities in noncommutative geometry are in general much smaller than those of their commutative counterparts. One can also address the problem of quantum rigidity of the structure group of noncommutative principle bundles understood via Hopf Galois extensions.

Additional project The work in this thesis gave rise to another project in [36].

The internal description of geometry described in this thesis can also be accommodated by the use of Topos Theory with techniques from Synthetic Differential Geometry. In order to view an automorphism group of a space as an object in a category it is imperative to use a topos. This is because the category of commutative algebras is not closed, but the presheaf category based on the category of commutative algebras, which is a topos, is closed.

In this project we have developed a sheaf theory approach to toric noncommutative geometry which allows us to formalize the concept of mapping spaces between two toric noncommutative spaces. As an application we have studied the 'internalized' automorphism group of a toric noncommutative space and shown that its Lie algebra has an elementary description in terms of braided derivations.

We consider the Gros Topos of sheaves of H-module (finitely presented) commutative algebras with a Zariski topology where H is the Hopf algebra on the torus and exhibit a fully faithful embedding of the internal derivations into a suitable notion of tangent bundle in the functor of points using the techniques of synthetic differential geometry. This generalised notion of a tangent bundle in the functor of points can thus be seen as the global version of the infinitesimal diffeomorphisms of toric noncommutative spaces. The topos perspective could offer an interpretation for physics for what a noncommutative space is via an understanding of how various types of structures map into it.

Appendix A

Category Theory

All definitions in this chapter are standard and can be found in [50, 8].

A.1 Categories, subcategories and isomorphism

Definition A.1.1 (Category)**.** A *category* $\mathscr{C}$ consists of a collection of objects $\mathrm{Ob}(\mathscr{C})$ together with a collection of morphisms $\mathrm{Morph}(\mathscr{C})$. $\mathrm{Morph}(\mathscr{C})$ is specified in terms of pairs of objects in $\mathrm{Ob}(\mathscr{C})$ which can be identified using a source $s : \mathrm{Morph}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ and target $t : \mathrm{Morph}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$. In this thesis, the following notation is used: For any two objects V, W in $\mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(V, W)$ is the collection of elements f of $\mathrm{Morph}(\mathscr{C})$ with $s(f) = V$ and $t(f) = W$. Then $\mathrm{Morph}(\mathscr{C})$ is the class of all $\mathrm{Hom}_{\mathscr{C}}(V, W)$ such that $V, W \in \mathrm{Ob}(\mathscr{C})$. In the case that $V = W$ the notation $\mathrm{End}_{\mathscr{C}}(V)$ is used. $\mathrm{Hom}_{\mathscr{C}}(V, W)$ may be empty for (some) $V \neq W \in \mathrm{Ob}(\mathscr{C})$, however $\mathrm{End}_{\mathscr{C}}(V)$ contains at least the identity morphism id_V for all $V \in \mathrm{Ob}(\mathscr{C})$. The constituents $\mathrm{Hom}_{\mathscr{C}}(V, W)$ of $\mathrm{Morph}(\mathscr{C})$ are referred to as Hom-classes. For each pair of Hom-classes for which the source of all morphisms in one coincides with the target of all morphisms in the other, there is defined a composition law $\circ$: $\mathrm{Hom}_{\mathscr{C}}(W, Z) \times \mathrm{Hom}_{\mathscr{C}}(V, W) \to \mathrm{Hom}_{\mathscr{C}}(V, Z)$, for example, and the composition law is associative. It will sometimes be convenient to use the short-hand notation $\mathscr{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{C}_0$ to denote a category, where $\mathscr{C}_0 := \mathrm{Ob}(\mathscr{C})$ and $\mathscr{C}_1 := \mathrm{Morph}(\mathscr{C})$.

Definition A.1.2 (Subcategory)**.** A category $\mathscr{B}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{B}_0$ is a *subcategory* of another category $\mathscr{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{C}_0$ if $\mathscr{B}_0 \subset \mathscr{C}_0$ and $\mathscr{B}_1 \subset \mathscr{C}_1$ with the same identity morphisms and composition of morphisms.

Definition A.1.3 (Full subcategory)**.** A category $\mathscr{B}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{B}_0$ is a *full subcategory* of a category $\mathscr{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{C}_0$ if $\mathscr{B}_0 \subset \mathscr{C}_0$ and if for any $f \in \mathscr{C}_1$ we have $s(f) \in \mathscr{B}_0$ then

$t(f) \in \mathscr{B}_0$ also.

All categories mentioned in this thesis are locally small.

Definition A.1.4 (Locally small)**.** A *locally small* category is a category $\mathscr{C}$ for which $\mathrm{Hom}_{\mathscr{C}}(V, W)$ is a set for all $V, W \in \mathscr{C}_0$. In this case the constituents $\mathrm{Hom}_{\mathscr{C}}(V, W)$ of $\mathscr{C}_1$ are referred to as Hom-sets. $\mathscr{C}$ is a small category if in addition $\mathscr{C}_0$ is a set.

Definition A.1.5 (Isomorphism)**.** Let $\mathscr{C}$ be a category. $V, W \in \mathscr{C}_0$ are *isomorphic* if there are morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(V, W)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(W, V)$ such that $f \circ g = \mathrm{id}_W$ the identity morphism on W and $g \circ f = \mathrm{id}_V$ the identity morphism on V.

The following categories feature in this thesis:

Example A.1.6 (Sets)**.** The category Set has the role as the category underlying all other categories in this thesis with $\mathrm{Ob}(\mathsf{Set})$ sets and $\mathrm{Morph}(\mathsf{Set})$ maps between sets. Set also features as a category in itself.

Example A.1.7 (Modules)**.** The category $\mathscr{M} := \mathrm{Mod}_k$ is the category of modules over a ring or field k.

Example A.1.8 (Algebras)**.** The category Alg is the category of algebras over the ring or field k.

Example A.1.9 (Bimodules)**.** For a given algebra A over k, the category $\mathrm{Bimod}(A)$ is the category of bimodules over A.

A.2 Functors, natural transformations and functor categories

Definition A.2.1 (Functor)**.** Given two categories $\mathscr{B}, \mathscr{C}$, a *functor* from $\mathscr{B}$ to $\mathscr{C}$ denoted by $F : \mathscr{B} \to \mathscr{C}$ assigns to an object $V \in \mathscr{B}_0$ an object $F(V) \in \mathscr{C}_0$, and to a morphism $f : V \to W$ in $\mathscr{B}_1$ a morphism $F(f) : F(V) \to F(W)$ in $\mathscr{C}_1$ in such a way that $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$ and $F(f \circ g) = F(f) \circ F(g)$ for any composable morphisms f, g in $\mathscr{B}_1$. We say that the assignment of objects $F(V)$ in $\mathscr{B}$ to objects V in $\mathscr{C}$ is *functorial* if F is a functor.

Objects in a category can be endowed with additional structure. A way to keep track of this is to use the forgetful functor:

Definition A.2.2 (Forgetful functor)**.** Suppose $\mathscr{B}$ and $\mathscr{C}$ are categories such that $\mathscr{B}_0$ is equal to $\mathscr{C}_0$ but has some additional structure. Then there is a *forgetful functor* $\mathrm{Forget} : \mathscr{B} \to \mathscr{C}$ which 'forgets' this additional structure.

Remark A.2.3. That Set is the locally small category underlying all categories means that there is a forgetful functor from all these categories to the category Set. That is there are forgetful functors $\mathrm{Forget} : \mathscr{M} \to \mathsf{Set}$, $\mathrm{Forget} : \mathrm{Alg} \to \mathsf{Set}$ and $\mathrm{Forget} : \mathrm{Bimod}(A) \to \mathsf{Set}$.

Remark A.2.4. Since functors preserve compositions (cf. Definition A.2.1) they preserve commutative diagrams.

Definition A.2.5 (Equivalence of functors)**.** Given two categories $\mathscr{B}$ and $\mathscr{C}$. Two functors $F, G : \mathscr{B} \to \mathscr{C}$ are said to be *equivalent* if

$$F(V) \cong G(V) \ , \tag{A.2.1}$$

are isomorphic as objects in $\mathscr{C}$ for all $V \in \mathscr{B}_0$.

Several functors have domain in the product of categories:

Definition A.2.6 (Product category)**.** Given two categories $\mathscr{B}, \mathscr{C}$, the *product category* $\mathscr{B} \times \mathscr{C}$ is the category whose objects are pairs (V, X) in the product of sets $\mathscr{B}_0 \times \mathscr{C}_0$ and whose morphisms are pairs (f, g) in the product of sets $\mathscr{B}_1 \times \mathscr{C}_1$.

Definition A.2.7 (Opposite category)**.** Given a category $\mathscr{C}$, the *opposite category*, which we denote by $\mathscr{C}^{\mathrm{op}}$, is defined as follows: the objects in $\mathscr{C}^{\mathrm{op}}$ are the same as the objects in $\mathscr{C}$ and the morphisms in $\mathscr{C}^{\mathrm{op}}$ are the morphism in $\mathscr{C}$ with reversed arrows; explicitly, a $\mathscr{C}^{\mathrm{op}}$-morphism $f^{\mathrm{op}} : V \to W$ is a $\mathscr{C}$-morphism $f : W \to V$ and the composition with another $\mathscr{C}^{\mathrm{op}}$-morphism $g^{\mathrm{op}} : W \to X$ is $g^{\mathrm{op}} \circ^{\mathrm{op}} f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}} : V \to X$.

Definition A.2.8 (Natural transformation)**.** Let $\mathscr{B}$ and $\mathscr{C}$ be categories and $F, G : \mathscr{B} \to \mathscr{C}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ is a collection of morphisms $\{\alpha_V : F(V) \to G(V)\}_{V \in \mathscr{B}}$ in $\mathscr{C}$ such that for any morphism $f : V \to V'$ in $\mathscr{B}$ the diagram

$$\begin{array}{ccc} F(V) & \xrightarrow{\alpha_V} & G(V) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(V') & \xrightarrow{\alpha_{V'}} & G(V') \end{array} \tag{A.2.2}$$

in $\mathscr{C}$ commutes. α is said to be a natural isomorphism if $(\alpha_V)^{-1}$ exists for each $V \in \mathscr{B}$.

Definition A.2.9 (Equivalence of categories)**.** Two categories $\mathscr{B}$ and $\mathscr{C}$ are said to be *equivalent* if there is a functor $F : \mathscr{B} \to \mathscr{C}$ and a functor $G : \mathscr{C} \to \mathscr{B}$ such that $F \circ G \Rightarrow 1_{\mathscr{C}}$ and $G \circ F \Rightarrow 1_{\mathscr{B}}$ are natural isomorphisms.

Definition A.2.10 (Functor category)**.** Given two categories $\mathscr{B}$ and $\mathscr{C}$ the *functor category* $[\mathscr{B}, \mathscr{C}]$ is the category whose objects are functors from $\mathscr{B} \to \mathscr{C}$ and whose morphisms are natural transformations between functors from $\mathscr{B} \to \mathscr{C}$. Natural transformations are composed associatively and there is an identity natural transformation from any functor to itself.

Definition A.2.11 (Comma category)**.** Given categories $\mathscr{B}, \mathscr{C}, \mathscr{D}$ and two functors $F : \mathscr{B} \to \mathscr{C}$, $G : \mathscr{D} \to \mathscr{C}$ the *comma category* $(F \Rightarrow G)$ has as objects triples (V, h, Y) where $V \in \mathscr{B}, Y \in \mathscr{D}$ and $h : F(V) \to G(Y)$ is a $\mathscr{C}$-morphism, and as morphisms $((V, h, Y) \to (V', h', Y'))$ pairs $(f : B \to B'\,,\, g : D \to D')$ in $\mathscr{B} \times \mathscr{D}$ such that

$$\begin{array}{ccc} F(V) & \xrightarrow{F(f)} & F(V') \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ G(Y) & \xrightarrow{G(g)} & G(Y') \end{array} \tag{A.2.3}$$

commutes in $\mathscr{C}$.

Definition A.2.12 (Slice category)**.** Given a category $\mathscr{C}$ and an object $W \in \mathscr{C}$ the *slice category* $(\mathscr{C} \Rightarrow W)$ has as objects pairs (V, h) where $V \in \mathscr{C}$ and $h : V \to W$ is a $\mathscr{C}$-morphism, and as morphisms $((V, h) \to (V', h'))$ a $\mathscr{C}$-morphism $f : V \to V'$

such that

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "h"'] & & V' \arrow[dl, "h'"] \\ & W & \end{tikzcd} \tag{A.2.4}$$

commutes in $\mathscr{C}$.

A.3 Monoids and monoidal categories

The concept of monoid appears in various forms throughout this thesis:

Definition A.3.1 (Monoid)**.** A *monoid* is an algebraic structure with a single associative binary operation and an identity element.

Example A.3.2. The Hom-set of a small category with a single object, equipped with the structure of its unit morphism (unit object) and (associative) composition of morphisms is a monoid. A functor is thus a monoid map on each Hom-set of a category.

Definition A.3.3 (Group)**.** A *group* is a monoid in which every element is invertible.

Remark A.3.4. In light of example A.3.2, one can model a group as a one object category in which all the morphisms are invertible.

Definition A.3.5 (Monoidal category)**.** A *monoidal category* $\mathscr{C} = (\mathscr{C}, \otimes, \Phi, I, \lambda, \varrho)$ consists of the following data: a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the tensor product, a natural isomorphism $\Phi : (\otimes \times \mathrm{id}) \circ \otimes \Rightarrow (\mathrm{id} \times \otimes) \circ \otimes$ called the assoiator and an object $I \in \mathscr{C}$ together with natural isomorphisms $\lambda : I \otimes \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$, $\varrho : \mathrm{id}_{\mathscr{C}} \otimes I \Rightarrow \mathrm{id}_{\mathscr{C}}$ called the unit object. The natural isomorphisms Φ, λ, ϱ are required to satisfy the following coherence conditions: the pentagon relations

$$\begin{tikzcd}
 & \big((Z \otimes Y) \otimes X\big) \otimes W \arrow[dl, "\Phi\otimes\mathrm{id}"'] \arrow[dr, "\Phi"] & \\
\big(Z \otimes (Y \otimes X)\big) \otimes W \arrow[d, "\Phi"] & & (Z \otimes Y) \otimes (X \otimes W) \arrow[d, "\Phi"] \\
Z \otimes \big((Y \otimes X) \otimes W\big) \arrow[rr, "\mathrm{id}\otimes\Phi"] & & Z \otimes \big(Y \otimes (X \otimes W)\big)
\end{tikzcd} \tag{A.3.1}$$

commutes for all $Z, Y, X, W \in \mathscr{C}$, and the triangle relations

$$\begin{tikzcd}
(I \otimes Y) \otimes X \arrow[rr, "\Phi"] \arrow[dr, "\lambda\otimes\mathrm{id}"'] & & I \otimes (Y \otimes X) \arrow[dl, "\lambda"] \\
 & Y \otimes X &
\end{tikzcd} \tag{A.3.2}$$

and

$$\begin{tikzcd}
(Y \otimes X) \otimes I \arrow[rr, "\Phi"] \arrow[dr, "\varrho\otimes\mathrm{id}"'] & & Y \otimes (X \otimes I) \arrow[dl, "\varrho"] \\
 & Y \otimes X &
\end{tikzcd} \tag{A.3.3}$$

commute for all $Y, X \in \mathscr{C}$. Due to Maclane's coherence theorem, the pentagon relations are sufficient to ensure that any pair of morphisms constructed as a sequence of associators from an object which all brackets collected on the left hand side $(((\cdots(V_1 \otimes V_2) \otimes \cdots) \otimes V_{n-1}) \otimes V_n)$ to an object with all brackets collected on the right hand side $(V_1 \otimes (V_2 \otimes (\cdots \otimes (V_{n-1} \otimes V_n) \cdots)))$, are equal.

Definition A.3.6 (Opposite tensor product)**.** Given a monoidal category $\mathscr{C}$, in addition to the monoidal functor $\otimes$ there is the functor describing the *opposite tensor product* $\otimes^{\mathrm{op}} := \otimes \circ \mathrm{flip} : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, where $\mathrm{flip} : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is the flip functor assigning to objects (V, W) in $\mathscr{C} \times \mathscr{C}$ the object $\mathrm{flip}(V, W) = (W, V)$ and to morphisms $\big(f : V \to X, g : W \to Y\big)$ in $\mathscr{C} \times \mathscr{C}$ the morphism $\mathrm{flip}\big(f : V \to X, g : W \to Y\big) = \big(g : W \to Y, f : V \to X\big)$

Definition A.3.7 (Braided monoidal category)**.** A *braided monoidal category* $\mathscr{C}$ is a monoidal category equipped with a braiding or commutativity constraint ψ which is a natural isomorphism $\psi : \otimes \Rightarrow \otimes^{\mathrm{op}}$, where $\otimes^{\mathrm{op}}$ is the opposite tensor product defined in Definition A.3.6, which satisfies the hexagon relations

$$\psi_{V\otimes W,Z} = \Phi_{Z,V,W} \circ (\psi_{V,Z} \otimes \mathrm{id}_W) \circ \Phi^{-1}_{V,Z,W} \circ (\mathrm{id}_V \otimes \psi_{W,Z}) \circ \Phi_{V,W,Z} \ , \tag{A.3.4a}$$

$$\psi_{V,W\otimes Z} = \Phi^{-1}_{W,Z,V} \circ (\mathrm{id}_W \otimes \psi_{V,Z}) \circ \Phi_{W,V,Z} \circ (\psi_{V,W} \otimes \mathrm{id}_Z) \circ \Phi^{-1}_{V,W,Z} \ , \tag{A.3.4b}$$

for all $V, W, Z \in \mathscr{C}$.

Definition A.3.8 (Strict (braided) monoidal category)**.** A *strict (braided) monoidal category* is a (braided) monoidal category for which the components of the associator and unitor (and braiding) natural transformations are the identity maps.

A.4 Limits and colimits

Given some objects and morphisms in a category $\mathscr{C}$, one may wish to form out of them another object which lies in $\mathscr{C}$. The notions of limit and colimit in a category give one such an object.

Definition A.4.1 (Diagram)**.** Let $\mathscr{C}$ be a category and $\mathbf{S}$ a small category. A *diagram of shape* $\mathbf{S}$ in $\mathscr{C}$ is a functor $D : \mathbf{S} \to \mathscr{C}$.

Definition A.4.2 (Cone)**.** Let $\mathscr{C}$ be a category, $\mathbf{S}$ a small category and D a diagram of shape $\mathbf{S}$ in $\mathscr{C}$. A *cone on* D consists of an object $C \in \mathscr{C}$ together with a family of $\mathscr{C}$-morphisms

$$(C \xrightarrow{f_S} D(S))_{S\in\mathbf{S}} \ , \tag{A.4.1}$$

such that for each **S**-morphism $f : S \to T$ the following diagram in $\mathscr{C}$ commutes

$$\begin{tikzcd} & D(S) \arrow[dd, "D(f)"] \\ C \arrow[ur, "f_S"] \arrow[dr, "f_T"'] & \\ & D(T) \end{tikzcd} \tag{A.4.2}$$

Definition A.4.3 (Limit)**.** Let $\mathscr{C}$ be a category, $\mathbf{S}$ a small category and D a diagram of shape $\mathbf{S}$ in $\mathscr{C}$. A *limit of* D is a cone

$$(L \xrightarrow{p_S} D(S))_{S \in \mathbf{S}} \ , \tag{A.4.3}$$

in $\mathscr{C}$ such that for any other cone (A.4.1) there is a unique $\mathscr{C}$-morphism $u : C \to L$ such that

$$p_S \circ u = f_S \ , \tag{A.4.4}$$

for all objects $S \in \mathbf{S}$.

Remark A.4.4. From the definition it follows that a limit of a diagram is unique up to a unique isomorphism. Hence with an abuse of notation we shall simply refer to the object L in the definition above as the limit of the diagram D.

Example A.4.5 (Equaliser)**.** The *equaliser* of two $\mathscr{C}$-morphisms g_1 and g_2 in a category $\mathscr{C}$ is the limit of a diagram of the shape

$$\begin{tikzcd} a \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & b \end{tikzcd} \ . \tag{A.4.5}$$

So if D is a diagram of shape (A.4.5) in the category $\mathscr{C}$, then the limit of D is the equaliser of the $\mathscr{C}$-morphisms $D(g_1)$ and $D(g_2)$ which is an object $L \in \mathscr{C}$ together with two $\mathscr{C}$-morphisms i, j such that the following two diagrams in $\mathscr{C}$ commute

$$\begin{tikzcd} & L \arrow[dl, "i"'] \arrow[dr, "j"] & \\ D(a) \arrow[rr, shift left, "D(g_1)"] \arrow[rr, shift right, "D(g_2)"'] & & D(b) \end{tikzcd} . \tag{A.4.6}$$

This is usually depicted diagrammatically by

$$\begin{tikzcd} L \arrow[r, "i"] & D(a) \arrow[r, shift left, "D(g_1)"] \arrow[r, shift right, "D(g_2)"'] & D(b) \end{tikzcd} , \tag{A.4.7}$$

with the understanding that $D(g_1) \circ i = D(g_2) \circ i$.

Example A.4.6 (Equaliser in $\mathscr{M}$)**.** In the category of k-modules $\mathscr{M}$ equalisers are kernels. In other words, the equaliser of two $\mathscr{M}$-morphisms g_1 and g_2 is the kernel

$$L = \operatorname{Ker}\big(g_1 - g_2\big) . \tag{A.4.8}$$

Example A.4.7 (Pullback)**.** Let $\mathscr{C}$ be a category. The *pullback* of two $\mathscr{C}$-morphisms g_1 and g_2 is the limit of a diagram of the shape

$$\begin{tikzcd} & a \arrow[d, "g_1"] \\ b \arrow[r, "g_2"'] & c \end{tikzcd} \tag{A.4.9}$$

So if D is a diagram of shape (A.4.9) in the category $\mathscr{C}$, then the limit of D is the pullback of the $\mathscr{C}$-morphisms $D(g_1)$ and $D(g_2)$ which is an object $L \in \mathscr{C}$ together with three $\mathscr{C}$-morphisms p_1, p_2, g such that the following two diagrams in $\mathscr{C}$ commute

$$\begin{tikzcd} L \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "g"] & D(a) \arrow[d, "D(g_1)"] \\ D(b) \arrow[r, "D(g_2)"'] & D(c) \end{tikzcd} \tag{A.4.10}$$

This translates to the condition that $D(g_1) \circ p_1 = D(g_2) \circ p_2$ and hence in depicting a pullback the $\mathscr{C}$-morphism g is usually dropped.

Example A.4.8 (Pullback in $\mathscr{M}$)**.** In the category of k-modules $\mathscr{M}$ the concept of a pullback coincides with that of fibred product; the $\mathscr{M}$-morphisms p_1 and p_2 are the projections onto the first and second components of the product.

Definition A.4.9 (Cocone)**.** Let $\mathscr{C}$ be a category, $\mathbf{S}$ a small category and D a diagram of shape $\mathbf{S}$ in $\mathscr{C}$. A *cocone on* D consists of an object $C \in \mathscr{C}$ together with a collection of $\mathscr{C}$-morphisms

$$(D(S) \xrightarrow{f_S} C)_{S \in \mathbf{S}} \ , \tag{A.4.11}$$

such that for each $\mathbf{S}$-morphism $f : S \to T$ the following diagram in $\mathscr{C}$ commutes

$$\begin{tikzcd} D(S) \arrow[dd, "D(f)"'] \arrow[dr, "f_S"] & \\ & C \\ D(T) \arrow[ur, "f_T"'] & \end{tikzcd} \tag{A.4.12}$$

Definition A.4.10 (Colimit)**.** Let $\mathscr{C}$ be a category, $\mathbf{S}$ a small category and D a diagram of shape $\mathbf{S}$ in $\mathscr{C}$. A *colimit of* D is a cocone

$$(D(S) \xrightarrow{p_S} L)_{S \in \mathbf{S}} \ , \tag{A.4.13}$$

in $\mathscr{C}$ such that for any other cocone (A.4.11) there is a unique morphism $u : L \to C$ such that

$$u \circ p_S = f_S \ , \tag{A.4.14}$$

for all objects $S \in \mathbf{S}$.

Remark A.4.11. From the definition it follows that a colimit of a diagram is unique up to a unique isomorphism. Hence with an abuse of notation we shall simply refer to the object L in the definition above as the colimit of the diagram D.

Example A.4.12 (Coequaliser)**.** Let $\mathscr{C}$ be a category. The *coequaliser* of two $\mathscr{C}$-

morphisms g_1 and g_2 is the colimit of a diagram of shape (A.4.5) in $\mathscr{C}$. So if D is a diagram of shape (A.4.5) in the category $\mathscr{C}$, then the colimit of D is the coequaliser of the $\mathscr{C}$-morphisms $D(g_1)$ and $D(g_2)$ which is a $\mathscr{C}$-object L together with two $\mathscr{C}$-morphisms p, q such that the following two diagrams in $\mathscr{C}$ commute

$$\begin{tikzcd} D(a) \arrow[rr, shift left, "D(g_1)"] \arrow[rr, shift right, "D(g_2)"'] \arrow[dr, "p"'] & & D(b) \arrow[dl, "q"] \\ & L & \end{tikzcd} \tag{A.4.15}$$

This is usually depicted diagrammatically by

$$\begin{tikzcd} D(a) \arrow[r, shift left, "D(g_1)"] \arrow[r, shift right, "D(g_2)"'] & D(b) \arrow[r, "q"] & L \end{tikzcd} , \tag{A.4.16}$$

with the understanding that $q \circ D(g_1) = q \circ D(g_2)$.

Example A.4.13 (Coequaliser in $\mathscr{M}$)**.** In the category of k-modules $\mathscr{M}$ the concept of a coequaliser coincides with that of a quotient and the $\mathscr{M}$-morphism q is the quotient map.

Appendix B

Additional proofs and notes

B.1 Cochain twisting of quasi-Hopf algebras

We fill in the details of Remark 2.1.41: The structure maps of a quasi-Hopf algebra H twisted by a cochain twist F followed by a cochain twist G satisfy

$$\Delta_{G\,F}(\,\cdot\,) = G\,\Delta_F(\,\cdot\,)\,G^{-1} = G\,F\,\Delta(\,\cdot\,)\,F^{-1}\,G^{-1} = (G\,F)\,\Delta(\,\cdot\,)\,(G\,F)^{-1}\ . \tag{B.1.1}$$

Denoting by $\partial\,G := (1\otimes G)\cdot(\mathrm{id}_{H_F}\otimes\Delta_F)(G)$, $\partial\,F := (1\otimes F)\cdot(\mathrm{id}_H\otimes\Delta)(F)$ and by $\partial\,G^{-1} := (\Delta_F\otimes\mathrm{id}_{H_F})(G^{-1})\cdot(G^{-1}\otimes 1)$, $\partial\,F^{-1} := (\Delta\otimes\mathrm{id}_H)(F^{-1})\cdot(F^{-1}\otimes 1)$ we have

$$\begin{aligned}\partial\,G\cdot\partial\,F &= (1\otimes G)\cdot(\mathrm{id}_{H_F}\otimes\Delta_F)(G)\cdot(1\otimes F)\cdot(\mathrm{id}_H\otimes\Delta)(F)\\ &= (1\otimes G\,F)\cdot(\mathrm{id}_H\otimes\Delta)(G\,F)\ ,\end{aligned} \tag{B.1.2}$$

where in the second equality we used the property (2.1.108) and the fact that Δ is an algebra morphism. By a similar calculation we have

$$\partial\,G^{-1}\cdot\partial\,F^{-1} = (\Delta\otimes\mathrm{id}_H)((G\,F)^{-1})\cdot((G\,F)^{-1}\otimes 1)\ . \tag{B.1.3}$$

Hence

$$\begin{aligned}\phi_{G\,F} &= \partial\,G\cdot\phi_F\cdot\partial\,G^{-1}\\ &= \partial\,G\cdot\partial\,F\cdot\phi\cdot\partial\,F^{-1}\cdot\partial\,G^{-1}\\ &= (1\otimes G\,F)\cdot(\mathrm{id}_H\otimes\Delta)(G\,F)\cdot\phi\cdot(\Delta\otimes\mathrm{id}_H)((G\,F)^{-1})\cdot((G\,F)^{-1}\otimes 1)\ .\end{aligned}$$

Finally

$$\begin{aligned}
\alpha_{G\,F} &= S(G^{(-1)})\,\alpha_F\,G^{(-2)} \\
&= S(F^{(-1)}\,G^{(-1)})\,\alpha\,F^{(-2)}\,G^{(-2)} \\
&= S((G\,F)^{(-1)})\,\alpha\,(G\,F)^{(-2)}\ ,
\end{aligned} \tag{B.1.4}$$

and by a similar calculation $\beta_{G\,F} = (G\,F)^{(1)}\,\beta\,S((G\,F)^{(2)})$.

B.2 Proofs of H-equivariance

The naturality condition (or H-equivariance condition) for the associator Φ in (2.2.21) is satisfied since by the coassociativity condition (2.1.98b) (and the functoriality of representations)

$$\begin{aligned}
&\rho_V \otimes (\rho_W \otimes \rho_X)(h) \circ \Phi_{V,W,X} \\
&= \big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)\big((\mathrm{id}_H \otimes \Delta)\,\Delta(h)\big)\big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)(\phi) \\
&= \big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)\big((\mathrm{id}_H \otimes \Delta)\,\Delta(h) \cdot \phi\big) \\
&= \big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)\big(\phi \cdot (\Delta \otimes \mathrm{id}_H)\,\Delta(h)\big) \\
&= \big((\rho_V \otimes \rho_W) \otimes \rho_X\big)(\phi)\big(\rho_V \otimes (\rho_W \otimes \rho_X)\big)\big((\Delta \otimes \mathrm{id}_H)\,\Delta(h)\big) \\
&= \Phi_{V,W,X} \circ (\rho_V \otimes \rho_W) \otimes \rho_X(h)\ ,
\end{aligned} \tag{B.2.1}$$

while the pentagon relations for Φ follow from the 3-cocycle condition (2.1.98c)

$$
\begin{aligned}
&\Phi_{V,W,X\otimes Z}\circ\Phi_{V\otimes W,X,Z}\\
&=\big(\rho_V\otimes(\rho_W\otimes(\rho_X\otimes\rho_Z))\big)((\mathrm{id}_H\otimes\mathrm{id}_H\otimes\Delta)(\phi))\\
&\cdot\big((\rho_V\otimes\rho_W)\otimes(\rho_X\otimes\rho_Z)\big)((\Delta\otimes\mathrm{id}_H\otimes\mathrm{id}_H)(\phi))\\
&=\big(\rho_V\otimes(\rho_W\otimes(\rho_X\otimes\rho_Z))\big)((\mathrm{id}_H\otimes\mathrm{id}_H\otimes\Delta)(\phi)\cdot(\Delta\otimes\mathrm{id}_H\otimes\mathrm{id}_H)(\phi))\\
&=\big(\rho_V\otimes(\rho_W\otimes(\rho_X\otimes\rho_Z))\big)((1\otimes\phi)\cdot(\mathrm{id}_H\otimes\Delta\otimes\mathrm{id}_H)(\phi)\cdot(\phi\otimes 1))\\
&=\big(\rho_V\otimes(\rho_W\otimes(\rho_X\otimes\rho_Z))\big)(1\otimes\phi)\\
&\cdot\big(\rho_V\otimes((\rho_W\otimes\rho_X)\otimes\rho_Z)\big)((\mathrm{id}_H\otimes\Delta\otimes\mathrm{id}_H)(\phi))\\
&\cdot\big((\rho_V\otimes(\rho_W\otimes\rho_X))\otimes\rho_Z\big)(\phi\otimes 1)\\
&=(\mathrm{id}\otimes\Phi_{W,X,Z})\circ\Phi_{V,W\otimes X,Z}\circ(\Phi_{VW,X}\otimes\mathrm{id})\ .
\end{aligned}
\tag{B.2.2}
$$

The naturality conditions for the left unitor λ in (2.2.23) are satisfied because of the unital condition (2.1.98a) and the k-linearity of representations

$$
\rho_V(h)\circ\lambda_V(c\otimes v)=\rho_V(h)(c\,v)=c\,\rho_V(h)(v)\ , \tag{B.2.3}
$$

for any $c\in k, v\in V$ and $h\in H$, while

$$
\lambda_V\circ(\rho_I\otimes\rho_V)(\Delta(h))(c\otimes v)=\lambda_V(\epsilon(h_{(1)})\,c\otimes\rho_V(h_{(2)})\,v)=c\,\rho_V(h)(v)\ , \tag{B.2.4}
$$

for any $c\in k, v\in V$ and $h\in H$. And similarly for the right unitor. The triangle relations for the left and right unitors λ and ϱ follow from the counital condition

(2.1.98d).

$$\begin{aligned}(\mathrm{id}\otimes\varrho)\circ\Phi &= (\mathrm{id}_V\otimes\varrho_W)(\rho_V\otimes(\rho_W\otimes\rho_I))(\phi)\\ &= (\mathrm{id}_V\otimes\varrho_W)(\rho_V\otimes(\rho_W\otimes\mathrm{id}_k))(\mathrm{id}_H\otimes\mathrm{id}_H\otimes\epsilon)(\phi)\\ &= \mathrm{id}_V\otimes\varrho_W\\ &= \varrho_{V\otimes W}\ . \end{aligned}\tag{B.2.5}$$

And similarly for the left unitor.

B.3 Weak associativity of internal composition

Using items (i) and (ii) of Proposition 2.2.13 we have

$$\begin{aligned}&\zeta^{-1}\big(\bullet\circ(\bullet\otimes\mathrm{id})\big)\\ &=\mathrm{ev}\circ(\bullet\circ(\bullet\otimes\mathrm{id})\otimes\mathrm{id})\\ &=\mathrm{ev}\circ(\bullet\otimes\mathrm{id})\circ((\bullet\otimes\mathrm{id})\otimes\mathrm{id})\\ &=\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi\circ((\bullet\otimes\mathrm{id})\otimes\mathrm{id})\\ &=\mathrm{ev}\circ(\bullet\otimes\mathrm{id})\circ((\mathrm{id}\otimes\mathrm{id})\otimes\mathrm{ev})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\\ &=\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi\circ((\mathrm{id}\otimes\mathrm{id})\otimes\mathrm{ev})\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\\ &=\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ((\mathrm{id}\otimes\mathrm{id})\otimes\mathrm{ev})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\ ,\end{aligned}\tag{B.3.1}$$

and on the other hand (using also the H-equivariance of the internal composition)

$$
\begin{aligned}
&\zeta^{-1}\big(\bullet \circ (\mathrm{id}\otimes\bullet)\circ\Phi\big)\\
&= \mathrm{ev}\circ\big(\bullet\circ(\mathrm{id}\otimes\bullet)\big)\circ\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\bullet\otimes\mathrm{id})\circ(\mathrm{id}\otimes\bullet\otimes\mathrm{id})\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi\circ(\mathrm{id}\otimes\bullet\otimes\mathrm{id})\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ(\mathrm{id}\otimes\bullet\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes(\mathrm{ev}\circ(\bullet\otimes\mathrm{id}))\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\Phi)\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev}))\circ(\mathrm{id}\otimes\Phi)\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\circ(\Phi\otimes\mathrm{id})\\
&= \mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ((\mathrm{id}\otimes\mathrm{id})\otimes\mathrm{ev})\circ(\mathrm{id}\otimes\Phi)\circ(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\circ(\Phi\otimes\mathrm{id})\ .
\end{aligned}
\tag{B.3.2}
$$

These two equations agree because of the 3-cocycle condition (2.1.98c).

B.4 Hexagon relations

The second Hexagon relations in Subsection 2.2.8 follows from (2.1.103b) by the calculation

$$
\begin{aligned}
\tau_{V,W\otimes Z} &= (\rho_V\otimes\rho_W)\otimes\rho_Z(R_{21})\\
&= (\rho_V\otimes(\rho_W\otimes\rho_Z))[(\mathrm{id}\otimes\Delta)(R)]_{312}\\
&= ((\rho_V\otimes\rho_W)\otimes\rho_Z)\big[\phi^{-1}_{231}\,R_{13}\,\phi_{213}\,R_{12}\,\phi^{-1}_{123}\big]_{312}\\
&= ((\rho_V\otimes\rho_W)\otimes\rho_Z)(\phi^{-1}\,R_{32}\,\phi_{132}\,R_{31}\,\phi^{-1}_{312})\\
&= ((\rho_W\otimes\rho_Z)\otimes\rho_V)(\phi^{-1})\,(\rho_W\otimes(\rho_V\otimes\rho_Z))(R_{32})\\
&\quad((\rho_W\otimes\rho_V)\otimes\rho_Z)(\phi_{132})\,((\rho_V\otimes\rho_W)\otimes\rho_Z)(R_{31})\,((\rho_V\otimes\rho_W)\otimes\rho_Z)(\phi^{-1}_{312})\\
&= \Phi^{-1}_{W,Z,V}\circ(\mathrm{id}_W\otimes\tau_{V,Z})\circ\Phi_{W,V,Z}\circ(\tau_{V,W}\otimes\mathrm{id}_Z)\circ\Phi^{-1}_{V,W,Z}\ .
\end{aligned}
\tag{B.4.1}
$$

B.5 Differential operators

Here we show the details omitted in the proof of Proposition 3.2.8.

We note first that in the case of a trivial associator ($\Phi = 1 \otimes 1 \otimes 1$) and R-matrix ($R = 1 \otimes 1$)

$$[X \bullet Y, a]^{(k)} = \sum_{j=0}^{k} \binom{k}{j} [X, a]^{(j)} \bullet [Y, a]^{(k-j)} \ . \tag{B.5.1}$$

Since we are only interested in knowing when an instance of the multibracket is zero and since the arguments of the multibracket are modules over H it will suffice to use the above formula even in the nonassociative and noncommutative case. Now if $V \subset \mathrm{diff}^n(W)$ and $U \subset \mathrm{diff}^m(W)$ it follows from Lemma 3.2.7 that $[X, a]^{(j)} = 0$ for all $X \in V$ and all $n+1 \leq j \leq n+m+1$, and that $[Y, a]^{(n+m+1-j)} = 0$ for all $Y \in U$ and all $m+1 \leq n+m+1-j \leq n+m+1$, that is for all $0 \leq j \leq n$. This implies that $[X \bullet Y, a]^{(n+m+1)} = 0$ for all $X \in V$ and all $Y \in U$. Therefore, again by Lemma 3.2.7, it follows that $V \bullet U \subset \mathrm{diff}^{(n+m)}(W)$ where $V \bullet U := \{X \bullet Y : X \in V, Y \in U\}$.

B.6 Endomorphisms of the unit object

Lemma B.6.1. *Let ρ_A be a commutative algebra in H–$\mathrm{Alg}^{\mathrm{com}}$. The internal endomorphism algebra* $(\mathrm{end}_A(\rho_A), \bullet)$ *is isomorphic to the braided commutative algebra* (ρ_A, μ) *in the category H–$\mathrm{Alg}^{\mathrm{com}}$.*

Proof. We define the $[H, \mathscr{M}]$-morphism

$$\lambda : \rho_A \Longrightarrow \mathrm{end}_A(\rho_A) \ , \tag{B.6.1}$$

with single component

$$\lambda : A \longrightarrow \mathrm{end}_A(A) \ , \qquad a \longmapsto \widehat{l}_A(a) \ , \tag{B.6.2}$$

for all $a \in A$. $\lambda(a)$ is indeed in $\mathrm{end}_A(A)$ for all $a \in A$ since $\widehat{l}_A$ is an H–Alg-morphism by Lemma 2.3.8 with source an algebra in H–$\mathrm{Alg}^{\mathrm{com}}$ and hence $[\widehat{l}_A(a), a'] = \widehat{l}_A\big(a\, a' - \rho_A(R^{(2)})(a')\, \rho_A(R^{(1)})(a)\big) = 0$. λ has an inverse given by

$$\lambda^{-1} : \mathrm{end}_A(\rho_A) \Longrightarrow \rho_A \ , \tag{B.6.3}$$

with single component

$$\lambda^{-1} : \mathrm{end}_A(A) \longrightarrow A \ , \qquad L \longmapsto \mathrm{ev}(L \otimes 1_A) \ , \tag{B.6.4}$$

for all $L \in \mathrm{end}_A(A)$. We have that

$$\lambda^{-1} \circ \lambda = \mathrm{id}_A \ , \tag{B.6.5}$$

because $\mathrm{ev}(\widehat{l}_A(a) \otimes 1_A) = a$ for all $a \in A$ and on the other hand,

$$\lambda \circ \lambda^{-1}(L) = \widehat{l}_A(\mathrm{ev}(L \otimes 1_A)) = l_A \circ \big(\zeta(\mathrm{id}) \circ \zeta^{-1}(\mathrm{id})(L \otimes 1_A) \otimes \mathrm{id}\big) = L \ . \tag{B.6.6}$$

Hence

$$\lambda \circ \lambda^{-1} = \mathrm{id}_{\mathrm{end}_A(A)} \ . \tag{B.6.7}$$

This completes the proof. □

B.7 Derivations are differential operators of order one

Proposition B.7.1. *Let ρ_A be a braided commutative algebra in H–$\mathrm{Alg}^{\mathrm{com}}$. The differential operators of order 1 on ρ_A decomposes as follows*

$$\mathrm{diff}^1(\rho_A) = \mathrm{der}(\rho_A) \oplus \mathrm{diff}^0(\rho_A) \ . \tag{B.7.1}$$

Proof. Using Lemma 3.2.7 we note first that if $V \subset \mathrm{der}(A) \oplus \mathrm{diff}^0(A)$ and $X \in V$, then $[[X,a],b] = [\widehat{l}_A\big(\mathrm{ev}\big(X \otimes a\big)\big), b] = 0$ (recalling that $\mathrm{diff}^0(A) = \mathrm{end}_A(A) \cong A$) and therefore $V \subset \mathrm{diff}^1(A)$. Now to prove the other inclusion we shall see that if X is a differential operator of order 1, then X can be decomposed into the sum

$$X = \widetilde{X} + \widehat{l}_A\big(\mathrm{ev}\big(X \otimes_A 1_A\big)\big) \ , \tag{B.7.2}$$

where $\widetilde{X} \in \mathrm{der}(A)$ is a derivation. In other words we shall see that if X is a differential operator of order 1, then

$$\widetilde{X} := X - \widehat{l}_A\big(\mathrm{ev}\big(X \otimes_A 1_A\big)\big) \ , \tag{B.7.3}$$

is a derivation. Using Lemma 3.2.2 we note that

$$\big[\widetilde{X}, a\big] = \big[X - \widehat{l}_A\big(\mathrm{ev}\big(X \otimes_A 1_A\big)\big), a\big] = \big[X, a\big] \ . \tag{B.7.4}$$

and on the hand

$$\begin{aligned}
\widehat{l}_A\big(\mathrm{ev}\big(\widetilde{X} \otimes a\big)\big) &= \widehat{l}_A\big(\mathrm{ev}\big(X \otimes a\big)\big) - \widehat{l}_A\big(\mathrm{ev}\big(\widehat{l}_A\big(\mathrm{ev}\big(X \otimes_A 1_A\big)\big) \otimes a\big)\big) \\
&= \widehat{l}_A\big(\mathrm{ev}\big(X \otimes a\big)\big) - \widehat{l}_A\big(\mathrm{ev}\big(X \otimes_A 1_A\big)\, a\big) \\
&= \widehat{l}_A\big(\mathrm{ev}\big(X\, a \otimes 1_A\big)\big) - \widehat{l}_A\big((-1)^{|a||X|}(R^{(2)} \triangleright a)\, \big(R^{(1)} \triangleright \mathrm{ev}\big(X \otimes_A 1_A\big)\big)\big) \\
&= \widehat{l}_A\big(\mathrm{ev}\big(X\, a \otimes 1_A\big)\big) - \widehat{l}_A\big(\mathrm{ev}\big((-1)^{|a||X|}(R^{(2)} \triangleright a)\, (R^{(1)} \triangleright X) \otimes_A 1_A\big)\big) \\
&= \widehat{l}_A\big(\mathrm{ev}\big([X,a] \otimes 1_A\big)\big) \\
&= [X,a] \ ,
\end{aligned} \tag{B.7.5}$$

where in the third step we have used that the action of H on the unit in A is the trivial action and in the fourth step that the evaluation map is left A-linear. □

B.8 Diagrammatic cochain twisting

The use of diagrams for proving results in category theory is essential.

The coproduct and H_F-equivariance We saw in Chapter 4 by the calculation in (4.2.4) that the formula for the twisted coproduct $\Delta_F(\cdot) = F\,\Delta(\cdot)\,F^{-1}$ and the $\star$-product in a twisted algebra $\mathcal{F}(A)$ is such that $\star = \mu \circ F^{-1}$ is H_F-equivariant.

Twisting the monoidal structure. From the definition of the coherence map for the monoidal structure in (2.2.24) we can interpret the formula $\star = \mu \circ F^{-1}$ (cf. (4.2.2)) to mean that we use the inverse twist F^{-1} to "untwist" the tensor product in $[H_F, \mathscr{M}]$ to that in $[H, \mathscr{M}]$ so that we can use the multiplication for the "untwisted" algebra in $[H, \mathscr{M}]$. Hence the coherence map can be read as a formula for untwisting a tesnor product: $\mathcal{F}(V) \otimes_F \mathcal{F}(W) \stackrel{F^{-1}}{\to} \mathcal{F}(V \otimes W)$.

From this reasoning we infer the following strategy for deriving formulae in the category $[H_F, \mathscr{M}]$ from formulae in $[H, \mathscr{M}]$:

The strategy. Beginning with a tensor product object which is twisted, we untwist it with the inverse of the (or the appropriate number of) twist(s), then we perform the required operation in the untwisted category, and finally re-twist the structure with the (or the appropriate number of) twist(s).

Applying this strategy now to the braiding and associator in H_F we have:

Braiding and the R_F-matrix. The formula for R_F comes from the following diagram

$$\begin{tikzcd} A_F \otimes_F A_F \arrow[r, "\tau_F"] \arrow[d, "F^{-1}"'] & A_F \otimes_F A_F \\ A \otimes A \arrow[r, "\sigma"'] & A \otimes A \arrow[u, "F_{21}"'] \end{tikzcd} \tag{B.8.1}$$

i.e. $\tau_F = F_{21} \circ \sigma \circ F^{-1}$.

Associator and ϕ_F. The formula for ϕ_F comes from the following diagram

$$\begin{tikzcd}
(A_F \otimes_F A_F) \otimes_F A_F \arrow[rr, "\phi_F"] \arrow[d, "F^{-1}\otimes 1"'] & & A_F \otimes_F (A_F \otimes_F A_F) \\
(A \otimes A)_F \otimes_F A_F \arrow[d, "(\Delta\otimes 1)(F^{-1})"'] & & A_F \otimes_F (A \otimes A)_F \arrow[u, "1\otimes F"'] \\
(A \otimes A) \otimes A \arrow[rr, "\phi"] & & A \otimes (A \otimes A) \arrow[u, "(1\otimes\Delta)(F)"']
\end{tikzcd} \tag{B.8.2}$$

i.e. $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

The correct formula for the coherence map for the internal hom-structure can be derived using the same strategy:

Twisting internal homomorphisms. Since the quasi-Hopf algebra H acts via the adjoint representation on internal homomorphisms $\rho_{\hom(V,W)}(\mathrm{id} \otimes S)(\Delta(h))$ for $h \in H$, the correct coherence map for the internal hom-structure is

$\hom_F(\mathcal{F}(V), \mathcal{F}(W)) \xrightarrow{(\mathrm{id}\otimes S)(F^{-1})} \mathcal{F}(\hom(V, W))$.

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